Riley Groups and Caruso Semigroups

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Plan

Riley groups

(topology) 3 Examples from topology:
I. 2-bridge complements
II. right triangles
III. "standard" orbifolds

Theory: survey of discrete Riley groups, cusp groups

Caruso semigroups

(dynamics) Theory: Mobius semigroups and their Julia sets
Attractors

Examples: Caruso semigroups $\begin{cases} \text{in slice} \\ \text{at cusps} \end{cases}$

Open problems

~~Source of~~

Historical perspective:

From ~~mid~~ 60's to mid 70's there were many surprising developments in knot theory.

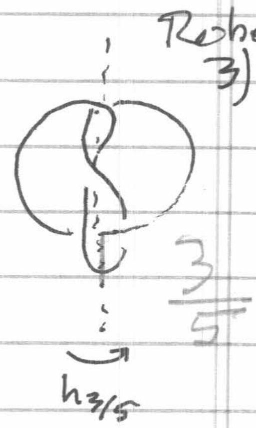
1) Conway (John Horton '69): Alexander polynomial has a skein relation!

~~Rational tangles~~

~~2-bridge knots/links are rational (tangles)~~

2) Stallings (~~Peter Ricketts~~ John '62): Many Seifert surfaces come from fibrations over the circle!

3) Riley (Robert '74): some knots are "hyperbolic"!



__Ex__ The figure-eight knot $K = 4_1$ (with 4 crossings) is hyperbolic in the sense that its complement $S^3 - K$ admits a Riemannian metric of __curvature__ __-1__ that is __complete__ and __finite volume__.

This prompted (Bill) Thurston's work on hyperbolization, showing __most__ knots K are hyperbolic, and geometrization.

$$\boxed{\partial H^3 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}}$$

More precisely: $\boxed{S^3 - (4_1) \cong H^3 / \langle z+\rho, \frac{z}{z+1} \rangle}$

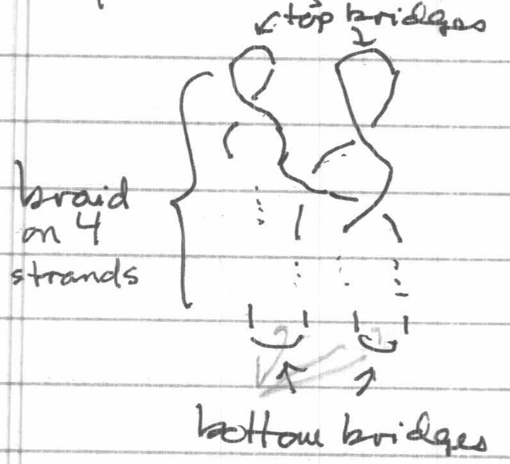
where $\rho = \exp\left(\frac{\pi i}{3}\right) = \sqrt[3]{-1}$.

Note that $z+\rho, \frac{z}{z+1}$ lie in the Bianchi group $PSl_2(\mathbb{Z}[\rho])$, which is discrete, so $\langle z+\rho, \frac{z}{z+1} \rangle$ is discrete. The $\cong$ is proven by comparing fundamental domains (as in Thurston's Princeton notes). Both generators are parabolic, fixing $\infty, 0$ resp.

Def $\langle z+a, \frac{z}{bz+1} \rangle$, $ab \neq 0$, is a Riley group

EXAMPLE I K a 2-bridge knot/link (such as 4_1)

if K comes from a braid on 4 strands by adding bridges on top and bottom



The two bottom meridians generate the group $\pi = \pi_1(S^3 - K)$, thanks to the Wirtinger presentation.

If K is hyperbolic then $\mathrm{vol}(S^3 - K) < \infty$ which implies meridians are parabolic.

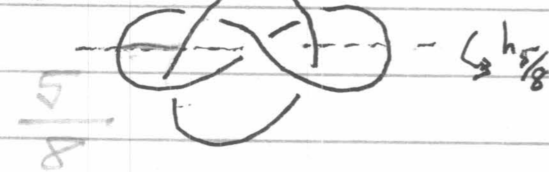
These generators can't commute, so in suitable coordinates $\hat{\mathbb{C}} \cong \partial H^3$ they fix ∞, 0.

So π is (essentially) a Riley group.

exp K = Whitehead link

$\pi \cong \langle z+\rho, \frac{z}{z+1} \rangle$

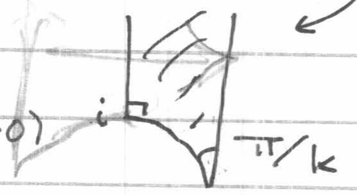
for $\rho = 1+i$.



Thm (Riley, based on Thurston) A 2-bridge knot/link is either toral (exception) OR hyperbolic (rule) (never both).

EXAMPLE II A hyperbolic right triangle T with one ideal vertex is isometric to T_k, some $k \geq 2$, in H^2

(vertices at ∞, i and in $\mathrm{Re}\, z > 0$)



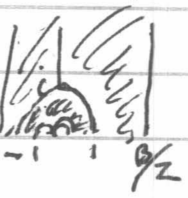
$\partial H^2 = \hat{\mathbb{R}}$

Let Γ_k be the reflection group in $\mathrm{Isom}(H^2)$ generated by the tile T_k. The orientation-preserving subgroup is a Hecke group

$$\Gamma_k^+ = \langle -1/z,\ z+\beta \rangle, \quad \beta = 2\cos(\pi/k) > 0$$

index 2 $\cup$

$$P_\beta = \langle z+\beta,\ \frac{z}{\beta z+1} \rangle, \text{ a Riley group with } \rho = \beta^2.$$



Hecke: $\langle -1/z, z+\beta \rangle$, $\beta > 0$, is discrete

$\iff$ $\beta = \underbrace{2\cos(\pi/k),\ k = 3, 4, \ldots}$, or $\underbrace{\beta \geq 2}$.

in which cases P_β has torsion or P_β is free.

Map

$\beta \in \mathbb{R}^+$ (PARAMETER RAY)

discrete groups with torsion

0 —— 1 —— $\sqrt{2}$ …… 2 ——→ free, discrete

(discrete, with torsion)

Note Hecke groups have dynamical value!

$\beta = 1$, $\Gamma_3^+ = PSl_2\mathbb{Z} = \left(\text{symmetry group of Farey tiling of } H^2\right)$, giving an elegant symbolic dynamics for billiard flow on T_3 (Felix Klein, Emil Artin, Carolyn Series)

which extends to $T_4, T_5, \ldots$ and much more

Plan mix the P/q in I. and the k in II. to get III.

~~the~~ ~~word~~ Parametrizing a 2-bridge knot/link

Fix P/q in lowest terms.

We'll define a dihedral group $D_{p/q} \subset O(4)$

so that $\underline{S^3/D_{p/q}} = S^3_{p/q}$ is an orbifold

whose underlying space is S^3 and

whose singular locus $L_{p/q}$ is a

2-bridge knot/link (with isotropy group $\mathbb{Z}_2$).
(as q odd/even)

In this way $\begin{pmatrix} \text{2-bridge knots/links} \\ \text{up to homeomorphism} \end{pmatrix} \longleftrightarrow \pm P/q \pmod 1$

Method Take $C_{p/q} = \{(\xi z_0, \xi^p z_1) : \xi^q = 1\} \subset U(2) \subset O(4)$
cyclic group of order q
and adjoin $(\overline{z_0}, \overline{z_1})$ to get $D_{p/q}$.

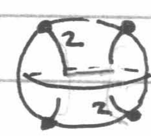
Then $(z_0, z_1) \in S^3 = T \cup B$ $\begin{pmatrix} T, B \\ \text{solid tori} \end{pmatrix}$

$\downarrow$ $\qquad |z_0| \leq |z_1| \qquad |z_1| \leq |z_0|$

Lens space $S^3/C_{p/q}$ = (solid torus) $\cup$ (solid torus)

$\downarrow$

$[z_0, z_1] \in S^3_{p/q}$ =  $T_{p/q}$ $\cup$  $B_{p/q}$

=  $\cup$ same

= (S^3 with singular locus $L_{p/q}$)

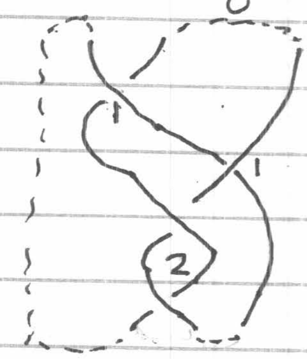
Note 1) the gluing map is isotopic to $id: S^2 \to S^2$ ~~fixing 4 points~~
and this isotopy gives the braid on 4-strands

2) 2-fold branched cover of S^3 branched at $L_{p/q}$
is the lens space $S^3/C_{p/q}$.

<u>Note</u> $L_{p/q}$ toral $\iff p \equiv \pm 1 \pmod q$
otherwise $L_{p/q}$ hyperbolic

<u>Note</u> Conway shows how to draw $L_{p/q}$

e.g. $p/q = 3/5 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}$ terms $1, 1, 2$ $\rightsquigarrow$ braid on 3 strands



which we fill in to get $L_{7/5}$ $(= 4_1)$.

$L_{5/8}$ = Whitehead link.

<u>EXAMPLE III</u> The "standard" orbifold $O(p/q, k)$
$k = 1, 2, 3, \ldots$

The involution $h(z_0, z_1) = (-z_0, z_1)$
$h : S^3 \to S^3$

commutes with $D_{p/q}$ and so induces an involution
$h_{p/q}([z_0, z_1]) = [-z_0, z_1]$
$h_{p/q} : S^3_{p/q} \to S^3_{p/q}$

stabilizing ~~fixing~~ the singular locus $L_{p/q}$.

The core circle $z_0 = 0$ in T gives a segment in $T_{p/q}$ joining the 2 singular arcs we denote $t_{p/q}$ (= <u>top tunnel</u>). Int $t_{p/q}$ lies in $(S^3_{p/q} - L_{p/q})$.

$O(p/q, k)$

$$\underset{\text{def}}{=} \begin{cases} (S^3_{p/q} - L_{p/q}) \text{ with Int } t_{p/q} \text{ labelled } k/2 & \left(\frac{k}{\text{even}}\right) \\ \text{OR} & \\ (S^3_{p/q} - L_{p/q})/h_{p/q} \text{ with Int } t_{p/q} \text{ labelled } k & \left(\frac{k}{\text{odd}}\right). \end{cases}$$

Suppose $O(p/q, k)$ is hyperbolic.
The orbifold group $\pi = \pi_1 O(p/q, k)$ is generated by the 2 bottom meridians, as before, so π is a Riley group.

Note ~~These~~ Standard orbifolds reconcile 2 approaches to ~~same~~ this problem:

"Classify discrete Riley groups that are not free".

Riley conjectured these groups lie in four families he called "Heckoid groups," which correspond to the four cases

q even/odd, k even/odd.

~~Ian~~ Agol gave orbifold approach ~~and~~ that may lead to a classification [but he missed the family q odd, k odd and so on]

What do we know?

Here is a converse to Riley's theorem

Thm (Colin Adams) each discrete Riley group that is torsionfree but not free is the group of a hyperbolic 2-bridge knot/link.

$k=2$ above

Conjecture a discrete Riley group with torsion is the group of a hyperbolic standard orbifold

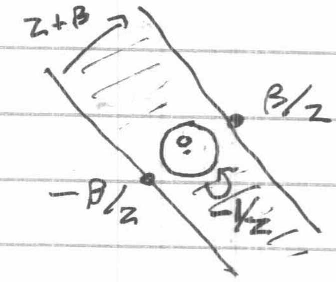
$k \neq 2$

(discrete + free?
not discrete?)

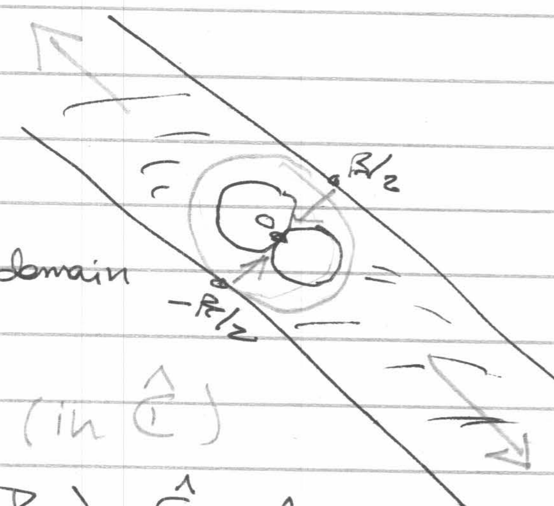
Discrete, free Riley groups

include P_β, $|\beta|>2$ $\quad$ ($|\rho|>4$)

Since in $\hat{\mathbb{C}}$ the extended group $\langle -1/z, z+\beta\rangle$ has a fundamental domain $|z|\geq 1$, $|\mathrm{Re}(z/\beta)|\leq \frac{1}{2}$



hence P_β has fundamental domain



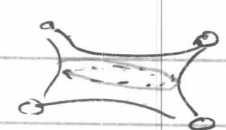
$\Rightarrow P_\beta$ free.

Recall limit set v.s. ordinary set

Let ~~Recall~~ Λ_β = (limit set of P_β) (in $\hat{\mathbb{C}}$)

Ω_β = (ordinary set of P_β) $= \hat{\mathbb{C}} - \Lambda_\beta$

and $\Sigma_\beta = \Omega_\beta / P_\beta$.

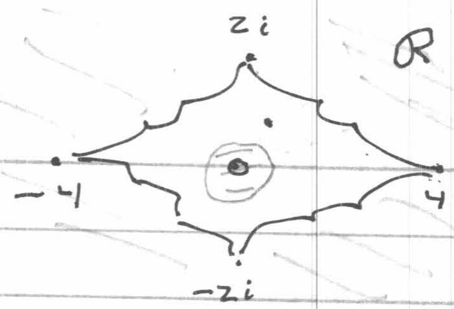
For $|\beta|>2$, Σ_β =  is a waisted 4-punctured sphere

with waist (corresponding to S') ~~the~~ isotopy class of simple curves that lift to ~~[illegible]~~ Ω_β.

Def The Riley slice $\mathcal{R} \subset \mathbb{C} - 0$ consists of all $\rho = \beta^2 \neq 0$ such that P_β is discrete, free and Σ_β is a 4-punctured sphere.

Thm (Lyubich + Suvorov) $\mathcal{R}$ is a topological annulus, indeed $\mathcal{R} = \mathcal{T}/\langle\tau\rangle$ where $\mathcal{T}$ is Teich space for $g=0$, $n=4$ and τ is Dehn twist around the waist.

$\mathcal{R}$ "large ρ" 9/13



In ρ − plane

($\rho = \beta^2$)

see Figure 2

contrast with ~~β~~ small ρ : if $|\beta| < 1$ then P_β is not discrete

(but $\beta = 1, \sqrt[3]{-1}$ give P_β discrete)
OR 1

(hard) Thm ~~(Maskit + Swarup)~~ P_β discrete, free $\iff \rho = \beta^2 \in \overline{\mathcal{R}}$.

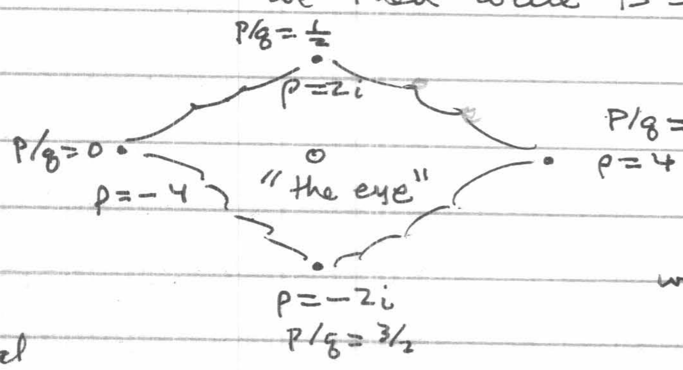
~~N~~Next we study $\partial\mathcal{R}$ (Jordan curve by Ohshika + Hideki Miyachi)
Ken'ichi

Bernard Maskit + Gadde Swarup showed for $\overset{\beta^2}{\overset{\shortparallel}{\rho}} \in \partial\mathcal{R}$ a dichotomy:

P_β is geometrically infinite, $\Lambda_\beta = \hat{\mathbb{C}}$

OR P_β is geometrically finite, Λ_β is a gasket and P_β has an accidental parabolic corresponding to a relator for some 2-bridge knot/link $L_{p/q}$

— we then write $\beta = C(p/q)$



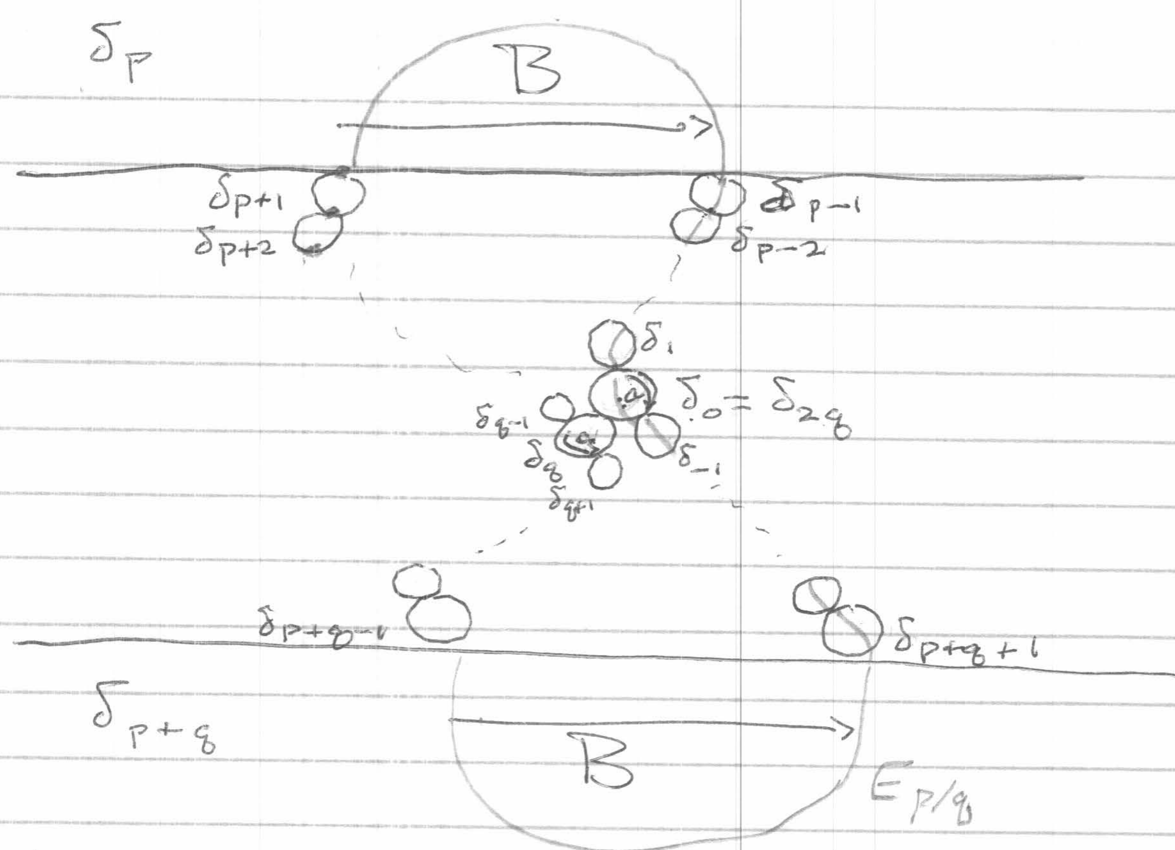
here $p/q \in \mathbb{Q}/2\mathbb{Z}$!

we say P_β ~~(cusp group)~~ is a cusp group in this case.

all our topological examples lie in the eye!
but at most points in the eye, P_β not discrete

see Examples in Figure 1 for
$$\begin{cases} \rho = \dfrac{3+\sqrt{7}i}{2} \\ p/q = 2/3 \end{cases}$$

$(p<q)$



David Wright, following Linda Keen + Carolyn Series explains the gasket Λ_c for $c = c(p/q)$

Here $\begin{cases} a(z) = \dfrac{z}{cz+1} \\ B(z) = z - c \end{cases}$ in P_c (except for a convenient rotation)

The contact points where δ_α touches $\delta_{\alpha+1}$ are fixed by accidental parabolics.

Caruso semigroups

$\pm\beta + \cfrac{1}{\pm\beta + \cfrac{1}{\pm\beta}}$

For fixed $\beta \in \mathbb{C}$, $\beta \neq 0$,

Caruso + Marotta

we iterate the maps $\pm\beta + \frac{1}{z}$, for all choices of sign.

z_0, $z_1 = \pm\beta + \frac{1}{z_0}$, $z_2 = \pm\beta + \frac{1}{z_1}$, ...
$z_0 = \infty$

When $|\beta| > 2$, z_n's ~~[illegible]~~ converge to a point in a Cantor set J'_β ! ~~[illegible]~~

We'll study the Caruso semigroup $S_\beta = \langle \underbrace{\beta + \tfrac{1}{z}}_{t(r(z))}, \underbrace{-\beta + \tfrac{1}{z}}_{t^{-1}(r(z))} \rangle$

where $r(z) = 1/z$, $t(z) = z + \beta$.
$(r^{-1} = r)$

Note our Riley group $P_\beta = \langle t, r^{-1} t r \rangle$ so P_β contains all even length words in the Caruso generators $\boxed{S_\beta^2 \subset P_\beta}$

dynamical interest

The Julia set J_β is the closure of the ~~hyperbolic~~ sources of elements of S_β

The dual Julia set J'_β is the closure of the ~~hyperbolic~~ sinks of elements of S_β

Both lie in Λ_β. $\left\{ \begin{array}{l} S_\beta(J'_\beta) \subset J'_\beta . \\ \cancel{S_\beta^{-1}(J_\beta)} \; J_\beta \subset S_\beta(J_\beta) \end{array} \right.$

EXAMPLE: when $\beta^2 \in \mathbb{R}$, Λ_β is a Cantor set and J_β, J'_β are disjoint (also Cantor sets)

All S_β-orbits approach J'_β in forward time

Under backward iteration, they approach J_β

J_β repeller

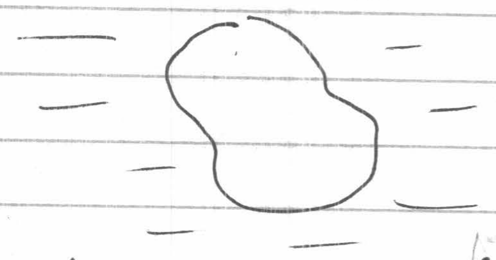
forward orbit

J'_β attractor

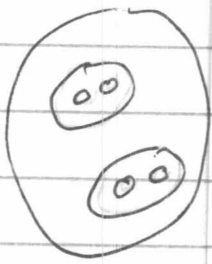
Proof Let E_β = "the equator"
be the shortest waist curve (in Poincaré metric on Σ_β)

$E_\beta = r(E_\beta)$ is disjoint from $\begin{cases} t(E_\beta) \\ t^{-1}(E_\beta) \end{cases}$

~~so~~ Define $N_\beta = \hat{\mathbb{C}} - (\text{interior of } E_\beta)$, so



N_β shaded
$\partial N_\beta = E_\beta$.



(*) N_β is compact, both $\boxed{(t^{\pm} r)}(N_\beta) \subset \text{int}(N_\beta)$
(and these are disjoint)

Now iterate, generate a Cantor set $= J'_\beta$ ✓

We say N_β is a block for S_β. Other β?

Thm S_β has an attractor / repeller pair
$\iff J_\beta, J'_\beta$ are disjoint
$\iff S_\beta$ admits a block N

EX $\beta = c(p/q) = c$.
Can define E_c (~~as~~ a limit of E_β, $\beta^2 \in \mathbb{R}$)
using Wright's $2q$-cycle of discs
The contact points A_c, $|A_c| = 2q$, satisfy
$$A_c \subset tr(A_c) \cup t^{-1}r(A_c)$$
Which shows $A_c \subset J'_c$. Likewise $A_c \subset J_c$,
so no attractor!
(Actually $A_c = J_c \cap J'_c$)

<u>EX</u> | $\beta^2 \in \partial Q \Longrightarrow$
| S_β has <u>no</u> attractor.

<u>Pf</u> ~~If~~ If S_β has an attractor, so does S_α for $\alpha \approx \beta$
(a block for S_β is a block for S_α, too)
But <u>cusps are dense</u> in ∂R and at a cusp one knows there is no attractor.

<u>Conj</u> for $\beta^2 \in$ eye, S_β has no attractor
(We know this when the P/q-relator is elliptic at β and we expect that these P/q-elliptic curves * are dense in the eye)

When P_β is discrete we know Λ_β and can surely compute J_β, J'_β. We should be able to understand ~~the~~ the dynamical system $J'_\beta \supseteq S_\beta$ using symbolic dynamics

<u>EX</u> P_β standard orbifold group, $k \geq 2$, we know $\Lambda_\beta =$ gasket $= J_\beta = J'_\beta$.

* such as 1/3-elliptic curve: trefoil relator

$$\operatorname{tr}(a\,b\,a\,b^{-1}a^{-1}b^{-1}) \in [-2,+2]$$
$$\| $$
$$2 - \rho - 2\rho^2 - \rho^3, \text{ cubic in } \rho$$

3/5-elliptic curve figure-eight relator

$$\operatorname{tr}(a b^{-1} a b\, a^{-1} b^{-1} a b^{-1} a^{-1} b) \in [-2,2]$$

= quintic in ρ, const. term 2