Spectral convergence in geometric quantization

Boston-Keio-Tsinghua workshop 2021

Kota Hattori (Keio)

- This talk is based on the joint work with Mayuko Yamashita (Kyoto Univ.)

1. Geometric quantization

Def. X : C^∞-manifold $(\dim = 2n)$

ω : 2-form on X

(X, ω) : symplectic manifold

$$:\Longleftrightarrow \begin{cases} \omega : \text{non-deg. } (\text{i.e. } \forall x,\ \omega_x^n \neq 0). \\ \omega : \text{closed } (\text{i.e. } d\omega = 0) \end{cases}$$

<u>Def.</u> (X, ω) : symplectic mfd.

(L, h, ∇) : prequantum bundle.

$:\Longleftrightarrow$

- $\pi : L \longrightarrow X$ is complex line bundle.
- h : hermitian metric on L.
- ∇ : hermitian connection on (L, h)

  s.t. $F^{\nabla} = -\sqrt{-1}\,\omega$.

- Geometric quantization

  --- define a "good" subspace

$$V \subset \{ s : \text{section of } L \to X \}.$$

- $V = C^\infty(L)$ --- contains too many sections.

$$\cup$$
$$V$$
$$\cup$$

- $V = \{ s \in C^\infty(L) \mid \nabla s \equiv 0 \} = \{0\}.$

  ($\because$ F^∇ is non-deg.)

By the additional structure,
(Polarization)

we can define "good" V s.t.

$$0 < \dim V < +\infty$$

Def. $P \subset TX \otimes \mathbb{C}$: polarization

$$:\Longleftrightarrow \begin{cases} \bullet\ P : \text{complex subbundle} \\ \bullet\ \omega|_P \equiv 0,\ [\Gamma(P), \Gamma(P)] \subset \Gamma(P) \\ \bullet\ \operatorname{rk} P = n. \\ \quad \left(\Leftrightarrow \forall x,\ \dim P_x = \frac{1}{2} \dim (T_x X \otimes \mathbb{C}) \right) \end{cases}$$

$$V_P := \{ s : \text{section of } L \mid \nabla_v s = 0 \ \forall v \in P \}.$$

Ex. (Kähler polarization)

J : ω-compatible complex structure on TX.

$$\rightsquigarrow \quad TX \otimes \mathbb{C} = \underset{\sqrt{-1}}{T^{1,0}X} \oplus \underset{-\sqrt{-1}}{T^{0,1}X}$$

$P_J := T^{0,1}X$: polarization.

$$\rightsquigarrow \quad V_{P_J} = H^0(L_J) \quad \left(\begin{array}{c} L_J \to X_J \\ \text{holo. line bundle} \end{array} \right)$$

Ex. (Real polarization)

$\mu : X \longrightarrow B$ (B : C^∞-mfd of $\dim = \frac{1}{2}\dim X$)

Assume
- μ : submersion ($d\mu_x$: surjective)
- $\forall b \in B$, $\mu^{-1}(b)$ is Lagrangian submfd ($\omega|_{\mu^{-1}(b)} \equiv 0$).

$P_\mu := (\mathrm{Ker}\, d\mu) \otimes \mathbb{C}$

$\rightsquigarrow$

$V_{P_\mu} = \{ s : \text{section of } L \mid \forall b,\ \nabla|_{\mu^{-1}(b)} s = 0 \}.$

<u>Def</u>. $\mu^{-1}(b)$ is a Bohr-Sommerfeld fiber

$:\Longleftrightarrow$ $\nabla|_{\mu^{-1}(b)}$ has a non-trivial parallel section

(if $\mu^{-1}(b)$ is connected, then $H^0(\nabla|_{\mu^{-1}(b)}) \cong \mathbb{C}$.)

Ex. (Bohr-Sommerfeld fiber)

$$X = \mathbb{R}^n \times T^n \ni (x_1, \cdots, x_n, e^{\sqrt{-1}y_1}, \cdots, e^{\sqrt{-1}y_n})$$

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

$L = X \times \mathbb{C}$. (trivial line bdle.)

$$\nabla = d - \sqrt{-1} \sum_i x_i dy_i$$

$\mu : X \longrightarrow \mathbb{R}^n$ (projection)

$\mu^{-1}(b)$: BS-fiber

$\Leftrightarrow b \in \mathbb{Z}^n$.

$$V_{P_\mu} = \bigoplus_{\mu^{-1}(b)\,:\,\text{BS fiber}} H^0(\nabla|_{\mu^{-1}(b)})$$

$$\cong \mathbb{C}^{\#\text{BS-fiber}} \quad \left(\text{if } {}^\forall \mu^{-1}(b) \text{ are connected}\right)$$

In many cases, we can observe

$$\dim V_{P_J} = \dim V_{P_{\ell}}.$$

(ex. abelian variety, toric variety, K3 surfaces).

Thm. (Baier - Florentino - Mourão - Nunes)

(X^{2n}, ω) : smooth toric variety

$\mu : X \to \Delta \subset \mathbb{R}^n$ moment map

delzant Polytope

$(\mu^{-1}(b) : BS \iff b \in \Delta \cap \mathbb{Z}^n)$

$(N = \#(\Delta \cap \mathbb{Z}^n))$

$\Rightarrow \exists \{J_s\}_{s>0}$: complex str. $\exists\, \sigma_1(s), \cdots, \sigma_N(s) \in H^0(L_{J_s})$

s.t. as $s \to 0$, $\sigma_\ell(s) \xrightarrow{s \to 0}$ δ-section supported by $\mu^{-1}(b_\ell)$

The similar phenomenon are observed in the case of

- Abelian variety (Baier-Mourão-Nunes)
- Flag manifold (Hamilton-Konno)

The aim of this talk :

Show

" $V_{P_{J_s}} \xrightarrow{s \to 0} V_{P_\mu}$ "

from the view point of $\begin{cases} \cdot \text{Spectral structures} \\ \cdot \text{Metric measure space.} \end{cases}$

2. Holomorphic sections and eigenfunctions.

(X, ω) : compact symplectic mfd.

(L, h, ∇) : prequantum bundle.

J : ω- compatible complex structure.

$$g_J := \omega(\cdot, J\cdot) \quad : \text{Riemannian metric.}$$

$$H^0(L_J) = \{ s \in C^\infty(L) \mid \nabla_{\bar\partial} s = 0 \}.$$

$$\left(\begin{array}{l} \nabla_{\bar\partial} : C^\infty(L) \longrightarrow \Omega^{0,1}(L). \\ \nabla_{\bar\partial}^* : \Omega^{0,1}(L) \longrightarrow C^\infty(L) : \text{formal ad.} \end{array} \right)$$

$$\Delta_{\bar\partial} := \nabla_{\bar\partial}^* \nabla_{\bar\partial} : C^\infty(L) \longrightarrow C^\infty(L).$$

X : cpt.

$$\Rightarrow \quad H^0(L_j) = \operatorname{Ker} \Delta_{\bar{\partial}}$$

$$\mathcal{S} := \{ u \in L \mid h(u,u) = 1 \}.$$

$$\begin{tikzcd} \mathcal{S} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

π : principal S^1-bundle.

Define $\hat{g}_J$: Riemannian met. on $\mathcal{S}$

as follows.

$$T_u P = H_u \oplus V_u.$$

horizontal

$V_u = T_u(\text{fiber of } P)$

Ker A_u.

$A \in \Omega^1(P, \sqrt{-1}\mathbb{R})$

$\updownarrow$: connection form.

∇

$$\hat{g}_J := \begin{array}{c c} \begin{array}{cc} H_u & \oplus \; V_u \end{array} & \\ \begin{pmatrix} g_J & 0 \\ 0 & A\otimes A \end{pmatrix} & \begin{array}{c} H_u \\ \oplus \\ V_u \end{array} \end{array}$$

$\rightsquigarrow$ $\hat{g}_J$: S^1-invariant Riemannian metric

$\Delta_{\hat{g}_J}$: Laplacian of $\hat{g}_J$.

$L^k = L^{\otimes k} \leftarrow$ hermitian met. & connection induced by (h, ∇).

$$\Delta_{\bar\partial} \circlearrowright C^\infty(L^k) \cong [C^\infty(S) \otimes \mathbb{C}]^{\rho_k} \circlearrowleft \Delta_{\hat g}$$

$$\left(\rho_k : \begin{array}{ccc} S^1 & \longrightarrow & \mathbb{C}^\times \\ \rotatebox{90}{$\in$} & & \\ \lambda & \longmapsto & \lambda^k \end{array} \right)$$

$$\Delta_{\hat g_J} : [C^\infty(S) \otimes \mathbb{C}]^{\rho_k} \longrightarrow [C^\infty(S) \otimes \mathbb{C}]^{\rho_k}$$

$$
\begin{tikzcd}
C^\infty(L^{\ell_k}) \arrow[r, phantom, "\cong"] \arrow[d, "\Delta_{\bar\partial}"'] \arrow[dr, phantom, "\circlearrowright"] & {[C^\infty(\mathbb{S}) \otimes \mathbb{C}]^{\rho_{\ell_k}}} \arrow[d, "\frac{\Delta_{\hat{g}_J} - (\ell_k^2 + n\ell_k)}{2}"] \\
C^\infty(L^{\ell_k}) \arrow[r, phantom, "\cong"] & {[C^\infty(\mathbb{S}) \otimes \mathbb{C}]^{\rho_{\ell_k}}}
\end{tikzcd}
$$

$$H^0(L_J^{k})$$

$$\parallel$$

0-eigenspace of $\Delta_{\bar\partial}$ $\cong$ $\dfrac{k^2+kn}{2}$-eigenspace of $\Delta_{\hat g_J}$

Convergence of $H^0(L_{J_s}^{k})$ $\Longleftrightarrow$ Conv. of eigenfct. of $\hat g_{J_s}$.

3. Spectral Convergence

(X, g) : Riemannian mfd.

$\rightsquigarrow$ d_g : distance fct.

μ_g : measure.

Δ_g : Laplacian.

(X, d, μ) : metric measure space

appearing as the "limit" of

Riemannian mfds $\{(X_i, g_i)\}_{i=0}^{\infty}$

measured Gromov-Hausdorff topology.

$\rightsquigarrow$ $\Delta_{d,\mu}$: Laplacian.

Spectral convergence

$\left(X_i, d_{g_i}, \frac{\mu_{g_i}}{A_i}\right)$: Riemannian mfds. $\left(\begin{array}{c} A_i > 0 \\ \text{normalization} \end{array}\right)$

$\Big\downarrow$ $i \to \infty$ (w.r.t. mGH topology)

(+ some assumptions)
(*)

(X, d, μ) : met. meas. sp.

$\Longrightarrow$ $\Delta_{g_i} \xrightarrow{i \to \infty} \Delta_{d,\mu}$.

- spectral convergence (Kuwae-Shioya).

$\mathcal{H}_i$: Hilbert spaces. (ex. $\mathcal{H}_i = L^2(X_i, \mu_{g_i})$)

$\Delta_i : \mathcal{D}(\Delta_i) \longrightarrow \mathcal{H}_i$ unbounded operator.

$\mathcal{D}(\Delta_i) \underset{\text{dense}}{\subset} \mathcal{H}_i$

(ex. $\Delta_i = \Delta_{g_i}$)

- Kuwae-Shioya defined $\begin{cases} \cdot\ \Delta_i \to \Delta_\infty \text{ strongly} \\ \cdot\ \Delta_i \to \Delta_\infty \text{ compactly.} \end{cases}$

- $\Delta_i \to \Delta_\infty$ compactly (All of spectra are supposed to be eigenvalues.)

$\Rightarrow$ $\exists \{\phi_{i,k}\}_{k=0}^{\infty}$: complete orthonormal system of H_i.

$\exists \lambda_{i,k} \geq 0$ s.t. $\Delta_i \phi_{i,k} = \lambda_{i,k} \phi_{i,k}$

and $\lim_{i\to\infty} \lambda_{i,k} = \lambda_{\infty,k}$, $\lim_{i\to\infty} \phi_{i,k} = \phi_{\infty,k}$

- $\Delta_i \to \Delta_\infty$ strongly

$\Rightarrow$ $\forall \lambda$: spectrum of Δ_∞ $\exists \lambda_i$: Spectrum of Δ_i s.t. $\lambda_i \to \lambda$.

(1) Fukaya.

(＊) $^\exists D, K > 0$

$\left.\begin{array}{l} \mathrm{diam}(X_i) \le D, \ |\text{sec. curv.}| \le K \end{array}\right\} \Rightarrow \underset{\text{cpt. conv.}}{\Delta_{g_i} \to \Delta_{d,\mu}}$

(2) Cheeger-Colding.

(＊) $^\exists D, K > 0$

$\left.\begin{cases} \mathrm{diam}(X_i) \le D \\ \mathrm{Ric}_{g_i} \ge -K g_i \end{cases}\right\} \Rightarrow \underset{\text{cpt. conv.}}{\Delta_{g_i} \to \Delta_{d,\mu}}$

(3) Kuwae - Shioya.

($\clubsuit$) $\exists K$ $\mathrm{Ric}_{g_i} \geq -K g_i$

(X_i are not necessarily ~~to~~ be cpt.)

$\Longrightarrow$ $\Delta_{g_i} \to \Delta_{d,\mu}$ (strong conv.)

Remind

$$H^0(L_J^k) \cong \frac{k^2+kn}{2}\text{-eigenspace of } \Delta_{\hat{g}_J}$$

Convergence of $H^0(L_{J_s}^k)$ $\Longleftrightarrow$ Conv. of eigenfct. of $\hat{g}_{J_s}$.

$\Uparrow$

Conv. of $(S, \hat{g}_{J_s})$ as $s \to 0$

4. Convergence of $(\mathcal{S}, \hat{g}_{J_s})$

Def. (X, d), (X', d') : metric spaces

$p \in X$, $p' \in X'$. $R, \varepsilon > 0$

$\phi : B_{d'}(p', R) \to X$: pointed (R, ε)-isometry

$$:\Longleftrightarrow \begin{cases} \cdot\ |d'(x, y) - d(\phi(x), \phi(y))| < \varepsilon \quad ({}^\forall x, y \in B(p', R)) \\ \cdot\ \phi(p') = p, \quad B_d(p, R-\varepsilon) \subset B(\mathrm{Im}\,\phi, \varepsilon) \end{cases}$$

Def. (X_i, d_i) : metric spaces, $p_i \in X_i$.

$$(X_i, d_i, p_i) \xrightarrow[\text{pGH}]{i \to \infty} (X_\infty, d_\infty, p_\infty)$$ (pointed Gromov-Hausdorff convergence)

$:\Longleftrightarrow$ $\exists R_i, \varepsilon_i > 0$ with $\lim_{i \to \infty} R_i = +\infty$, $\lim_{i \to \infty} \varepsilon_i = 0$

$\exists \phi_i : B(p_i, R_i) \longrightarrow X_\infty$ pointed (R_i, ε_i)-isometry

$\curvearrowleft S^1$ (isometrically)

Def. (X_i, d_i) : metric spaces, $p_i \in X_i$. S^1-equiv.

μ_i measure

$$(X_i, d_i, p_i) \xrightarrow[S^1\text{-pmGH}]{i\to\infty} (X_\infty, d_\infty, p_\infty)$$

measured

(pointed Gromov-Hausdorff convergence)

$:\iff \exists\, R_i, \varepsilon_i > 0$ with $\lim\limits_{i\to\infty} R_i = +\infty$, $\lim\limits_{i\to\infty} \varepsilon_i = 0$

$\exists\, \phi_i : B(p_i, R_i) \longrightarrow X_\infty$ pointed (R_i, ε_i)-isometry

Borel, S^1-equiv.

& Convergence of measures

Setting

- (X^{2n}, ω) : symplectic mfd.
- (L, h, ∇) : prequantum bundle.
- $\mu : X \longrightarrow B$ Lagrangian fibration,
- $b \in B$: regular value of μ.
- $\mathcal{S} = \mathcal{S}(L, h)$, $p \in \mathcal{S}|_{\mu^{-1}(b)}$ (base point).
- $\{J_s\}_{s>0}$: family of ω-compatible cpx structures

Def. (1) k : positive integer.

$\mu^{-1}(b)$: k-BS fiber

$:\Longleftrightarrow$ $\left(L^{k}|_{\mu^{-1}(b)}, \nabla\right)$ has a

nontrivial parallel section

(2) $\mu^{-1}(b)$: strict k-BS fiber

$:\Longleftrightarrow$ $\mu^{-1}(b)$: k-BS & not l-BS for any $l<k$

Assume.

(1) μ is proper, $\forall y \in B$ $\mu^{-1}(y)$ is connected

($\Rightarrow \mu^{-1}(y) \cong T^n$ by Liouville-Arnold thm)

(2) $\mathcal{P}_{J_s} \longrightarrow \mathcal{P}_\mu$ $(s \to 0)$

as polarization.

(& some technical condition).

- Local model of $P_{J_s} \xrightarrow{s\to 0} P_\mu$.

$$X = \mathbb{R}^n \times T^n, \quad \omega = \sum_{i=1}^n dx_i \wedge dy_i, \quad \mu : X \to \mathbb{R}^n$$

$$J_s : \quad \frac{\partial}{\partial x_i} \mapsto \frac{2\pi}{s}\frac{\partial}{\partial y_i}, \qquad \frac{\partial}{\partial y_i} \mapsto -\frac{s}{2\pi}\frac{\partial}{\partial x_i}$$

$$g_{J_s} = \frac{2\pi}{s}\sum_i dx_i^2 + \frac{s}{2\pi}\sum_i dy_i^2$$

Thm 1. (H.) $m > 0$: integer.

$\mu^{-1}(b)$: strict m-BS fiber, $p \in \mathfrak{S}|_{\mu^{-1}(b)}$

Then $\exists (\mathfrak{S}_{(m)}, d_{(m)}, \mu_{(m)})$ m.m.space

with isometric S^1-action and $\exists A > 0$ s.t.

$$\left(\mathfrak{S}, d_{\hat{g}_{J_s}}, \frac{A\mu_{\hat{g}_{J_s}}}{\sqrt{s}^{\,n}}, p\right) \xrightarrow[S^1\text{-pmGH}]{} \left(\mathfrak{S}_{(m)}, d_{(m)}, \mu_{(m)}, p_{(m)}\right)$$

can be described explicitly

Thm 2 (H1).

Let $\mu^{-1}(b)$ be not m-BS fiber for $\forall m$.

Then $\exists A > 0$

$$\left(\mathfrak{F},\ d_{\hat{g}_{J_s}},\ \frac{A\cdot\mu_{\hat{g}_{J_s}}}{\sqrt{s}^{\,n}},\ p\right) \xrightarrow[S^1\text{-pmGH}]{} \left(\mathbb{R}^n,\ d_{Euc},\ \mathcal{L}_{\mathbb{R}^n},\ 0\right)$$

Lebesgue measure

Euclidean metric

$$\left(\mathcal{S}_{(m)},\ d_{(m)},\ \mu_{(m)},\ p_{(m)} \right)$$

- $\mathcal{S}_{(m)} := S^1 \times \mathbb{R}^n \quad \left(\ni \left(e^{\sqrt{-1}t}, x_1, \dots, x_n \right) \right)$

- $\mathcal{S}_{(m)} \curvearrowleft S^1 \ : \ \left(e^{\sqrt{-1}t}, x \right) \cdot \lambda := \left(e^{\sqrt{-1}t} \cdot \lambda^m, x \right)$

- $g_{(m)} := \dfrac{dt^2}{m^2 (1 + \|x\|^2)} + \sum_{i=1}^{n} dx_i^2 \rightsquigarrow d_{(m)}$

- $\mu_{(m)} = dt\, dx_1 \cdots dx_n$
- $p_{(m)} = (1, 0)$

$\Delta_{(m)}$: weighted Laplacian of $(\mathcal{S}_{(m)}, d_{(m)}, \mu_{(m)})$

$\rightsquigarrow$

$$\Delta_{(m)} : [C^\infty(\mathcal{S}_{(m)}) \otimes \mathbb{C}]^{\rho_k} \longrightarrow [C^\infty(\mathcal{S}_{(m)}) \otimes \mathbb{C}]^{\rho_k}$$

$$\Vert \qquad (k > 0 : \text{integer})$$

$$\{0\} \quad (\text{if } k \notin m\mathbb{Z})$$

If $\hbar \in m\mathbb{Z}$, then

$$\Delta_{(m)} - (n\hbar + \hbar^2) : [C^\infty(S_{(m)}) \otimes \mathbb{C}]^{\rho_\hbar} \longrightarrow [C^\infty(S_{(m)}) \otimes \mathbb{C}]^{\rho_\hbar}$$

$$\cong$$

$$\Delta^\hbar_{\mathbb{R}^n} : C^\infty(\mathbb{R}^n) \otimes \mathbb{C} \longrightarrow C^\infty(\mathbb{R}^n) \otimes \mathbb{C}$$

$$:= \sum_{i=1}^n \left\{ -\left(\frac{\partial}{\partial x^i}\right)^2 + 2\hbar\, x_i \frac{\partial}{\partial x^i} \right\} : \text{Laplacian on } (\mathbb{R}^n, |dx|^2, e^{-\hbar \|x\|^2} dx_1 \cdots dx_n)$$

Thm. 3 (Yamashita - H.)

Let X be cpt, $\forall b \in B$: regular val. of μ,

$BS_{\hbar} := \{ b \in B \mid \mu^{-1}(b)$ is a $\hbar$-BS fiber$\}$.

(& Assumptions for J_s, μ).

(For example, X = abelian variety.)

$$\Longrightarrow \quad \Delta_{\bar{\partial}_{J_s}} \xrightarrow[s \to 0]{} \bigoplus_{b \in BS_{\hbar}} \frac{1}{2} \Delta^{\hbar}_{\mathbb{R}^n}$$

(compact convergence)

- [Yamashita-H.]

X : cpt smooth toric variety

$$\Rightarrow \quad \Delta_{\bar{\partial}_{J_s}} \xrightarrow[s\to 0]{\text{cpt.}} \bigoplus_{b \in BS_\ell} \frac{1}{2}\Delta^{\mathbb{R}}_{\mathbb{R}^n}(b)$$

depend on whether

b is in bdry of

Delzant polytope or not.

• [H1.]

Special Lagrangian fibration by hyperkähler rotation.

X : elliptic K3 surface.

with 24 singular fibers of type I_1.

$$\Rightarrow \quad \Delta_{\bar{\partial}_{J_s}} \xrightarrow[s \to 0]{} \bigoplus_{b \in BS_\varepsilon} \frac{1}{2} \Delta^{\mathbb{R}}_{\mathbb{R}^n}$$

indep. whether $\mu^{-1}(b)$ is singular or not.