Stability of non-proper functions

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$f : N \to P$: proper $:\Leftrightarrow \forall K \subset P$: compact, $f^{-1}(K)$: compact

A function is a C^∞-mapping to $\mathbb{R}$ (i.e. $P = \mathbb{R}$).

Assume that mfd's are C^∞, second countable & have no ∂.

§.1 Introduction

◇ Notations

- N, P : manifolds

  $C^\infty(N, P) := \{f : N \to P : C^\infty\text{-mapping}\}$

  We endow $C^\infty(N, P)$ with the "Whitney C^∞-topology"

  (Roughly speaking, two mappings are close to each other under this topology iff they have close differentials.)

- $\mathbf{Diff}(N) \subset C^\infty(N, N)$: set of self-diffeomorphisms

  $\mathbf{Diff}(N)$ is endowed with the relative topology

- For $f \in C^\infty(N, P)$, $C_f := \{x \in N \mid \mathrm{rank}(df_x) < \dim P\}$.

◇ Definition of stability

Definition

- $f, g \in C^\infty(N, P)$ are **right-left equivalent** ($f \sim g$)
  $:\Leftrightarrow \exists \Phi \in \mathrm{Diff}(N), \exists \phi \in \mathrm{Diff}(P)$ s.t. $g = \phi \circ f \circ \Phi$.
- $f \in C^\infty(N, P)$ is **stable** (w.r.t. the Whitney topology)
  $:\Leftrightarrow \exists \mathcal{U} \subset C^\infty(N, P)$: nbhd. of f (w.r.t. the Whitney topology)
  s.t. $\forall g \in \mathcal{U}$ is right-left equivalent to f.

It is in general difficult to check whether a given mapping is stable or not!!

◇ Simple examples (1/2)

$f_n \in C^\infty(\mathbb{R},\mathbb{R})$ defined by $f_n(x) := x^n$.

Claim 1. $f_1 = \mathrm{id}_{\mathbb{R}}$ is stable.

Proof : Define $\mathcal{U} := \left\{ g \in C^\infty(\mathbb{R},\mathbb{R}) \,\middle|\, \forall t \in \mathbb{R},\ g'(t) > \frac{1}{2} \right\}$. Then,

- $\mathcal{U}$ is an open nbhd. of f_1.
- By the inverse func., intermediate val. & mean val. theorems, $\mathcal{U} \subset \mathrm{Diff}(\mathbb{R})$, in particular $f_1 \sim g$ $(g = g \circ f_1 \circ \mathrm{id}_{\mathbb{R}})$ for $\forall g \in \mathcal{U}$.

Thus, f_1 is stable. □

◇ Simple examples (2/2)

Claim 2. f_3 (in general f_n for $n \geq 3$) is not stable.

Proof (for $n = 3$) : Define $g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ by $g_t(x) := x^3 + \rho(x)tx$, where $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$, $\rho(x) \equiv 1$ for $|x| \leq 1$, $\rho(x) \equiv 0$ for $|x| \geq 2$.

- For $0 < \forall t \ll 1$, g_t has no critical points, in particular $g_t \not\sim f_3$.
- $\mathbb{R} \ni t \mapsto g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ is continuous, and thus
  $\forall \mathcal{U} \subset C^\infty(\mathbb{R}, \mathbb{R})$: open nbhd. of f_3, $\exists t > 0$ s.t. $g_t \in \mathcal{U}$.

Thus, f_3 is not stable. □

How about f_2...? It is not so easy to show that it is stable...

◇ Definition of infinitesimal stability

$\Gamma(E)$: set of sections of a vector bundle E.

Definition

$f \in C^\infty(N,P)$ is **infinitesimally stable**

$:\Leftrightarrow \Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP))$, where

$df_* : \Gamma(TN) \to \Gamma(f^*TP)$ is defined by $df_*(\xi) := df \circ \xi$.

$f^* : \Gamma(TP) \to \Gamma(f^*TP)$ is defined by $f^*(\eta) := \eta \circ f$.

Remark (Motivation for infinitesimal stability)

$L_f : \mathrm{Diff}(N) \times \mathrm{Diff}(P) \to C^\infty(N,P)$, $L_f(\Phi,\phi) := \phi \circ f \circ \Phi^{-1}$.

- stability $\Leftrightarrow$ image of L_f contains a nbhd. of f.
- inf. stability $\Leftrightarrow$ the "differential $(dL_f)_{(\mathrm{id}_N,\mathrm{id}_P)}$" is surjective.

f : inf. stable :$\Leftrightarrow$ $\Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP))$.

$\diamondsuit$ **Simple examples** (again)

$f_n \in C^\infty(\mathbb{R}, \mathbb{R})$ defined by $f_n(x) := x^n$.

Claim 3. f_2 is infinitesimally stable.

Proof : We can identify $\Gamma(T\mathbb{R}) = \Gamma(f_2^*T\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R})$.

Under these identifications, $(df_2)_*(\xi) = 2x\xi$ and $f_2^*(\xi) = \xi(x^2)$.

Since $\xi(x) = \xi(0) + \int_0^1 \frac{d}{dt}(\xi(tx))\, dt = \xi(0) + x \int_0^1 \frac{d\xi}{dt}(tx)dt$

for $\xi \in C^\infty(\mathbb{R}, \mathbb{R})$, $\Gamma(f_2^*T\mathbb{R}) = (df_2)_*(\Gamma(T\mathbb{R})) + f_2^*(\Gamma(T\mathbb{R}))$. $\square$

◇ Stability for proper mappings (1/2)

Theorem (Mather 1970)

For $f \in C^\infty(N,P)$: **proper** mapping, stability, infinitesimal stability, strong stability and "local stability" are all equivalent.

Definition $f \in C^\infty(N,P)$: **strongly stable**

$:\Leftrightarrow \exists \mathcal{U} \subset C^\infty(N,P)$: neighborhood of f

$\exists(\Theta, \theta) : \mathcal{U} \to \mathrm{Diff}(N) \times \mathrm{Diff}(P)$: **continuous** map

s.t. $\forall g \in \mathcal{U}$, $\theta(g) \circ g \circ \Theta(g) = f$.

◇ Stability for proper mappings (2/2)

We will only give several properties of "local stability".

- local stability is the weakest condition of the four stabilities.

  i.e. (inf.) stable $\Rightarrow$ locally stable for general (possibly non-proper) f.

- In general, it is (relatively) easy to check local stability (Mather).

  e.g. $f : N \to \mathbb{R}$: (possibly non-proper) function is locally stable

  $\Leftrightarrow f$: Morse function, that is,

  – $\forall x \in C_f$, $\det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{i,j} \neq 0$.

  – $f|_{C_f}$: injective.

Thus, **it is easy to check stability of proper mappings!!**

◇ Motivating problem 1

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. Is $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by $f(x, y) = x^2 - y^2$ stable?

Note that f is infinitesimally stable but **NOT** strongly stable!!

(will be seen later)

◇ Remarks on problem 1 (1/2)

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. $f(x, y) = x^2 - y^2$: stable?

- f : inf. stable $\Leftrightarrow$ f : loc. stable & $f|_{C_f}$: proper (Mather).

  In particular, infinitesimal stability is easily checked.

  (since it is easy to check local stability.)

  However, it is in general difficult to check (strong) stability!

◇ Remarks on problem 1 (2/2)

Problem 1

How can we detect (strong) stability of non-proper functions?

e.g. $f(x, y) = x^2 - y^2$: stable?

- (Dimca) $f \in C^\infty(\mathbb{R}, \mathbb{R})$: stable

  $\Leftrightarrow f$: locally stable & $f(C_f) \cap (\mathcal{S}(f) \cup L(f)) = \emptyset$, where

$$L(f) = \left\{ y \in \mathbb{R} \;\middle|\; y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\}$$

$$\mathcal{S}(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \;\middle|\; \{x_i\} : \begin{array}{l} \text{sequence in } C_f \text{ without} \\ \text{accumulation points} \end{array} \right\}$$

Thus, it is (somewhat) easy to check stability of $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

$f \in C^\infty(\mathbb{R}, \mathbb{R})$: stable $\Leftrightarrow$ f : locally stable & $f(C_f) \cap (\mathcal{S}(f) \cup L(f)) = \emptyset$.

$L(f) = \left\{ \lim_{x \to \pm\infty} f(x) \right\}$, $\mathcal{S}(f) = \left\{ \lim_{i \to \infty} f(x_i) \;\middle|\; \{x_i\} : \begin{array}{l} \text{seq. in } C_f \text{ w/o} \\ \text{accumulation pt's} \end{array} \right\}$

Example $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \exp(x) \sin x$.

Since $f^{(k)}(x) = 2^{k/2} \exp(x) \sin\left(x + \dfrac{k\pi}{4}\right)$, it is easy to see:

- $C_f = \left\{ \dfrac{(4n+3)\pi}{4} \in \mathbb{R} \;\middle|\; n \in \mathbb{Z} \right\}$,
- f : Morse func. (i.e. $f|_{C_f}$: inj. & $\forall x \in C_f$, $f^{(2)}(x) \neq 0$).

Furthermore, $\mathcal{S}(f) = L(f) = \{0\}$ & $0 \notin f(C_f) \Rightarrow f$: stable

On the other hand, $(f|_{C_f})^{-1}([-1, 1])$: infinite discrete set

$\Rightarrow f$: NOT infinitesimally stable ($\because f|_{C_f}$: not proper).

$\diamondsuit$ Motivating problem 2

Problem 2

How are the four stabilities related for non-proper functions?

In particular, strongly stable $\Rightarrow$ infinitesimally stable?

◇ Remarks on problem 2 (1/3)

Problem 2

How are the four stabilities related for non-proper functions?

In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- f : strongly stable $\Rightarrow$ f : stable (obvious).
- f : stable $\Rightarrow$ f : locally stable (Mather).
- f : inf. stable $\Leftrightarrow$ f : loc. stable & $f|_{C_f}$: proper (Mather).

◇ Remarks on problem 2 (2/3)

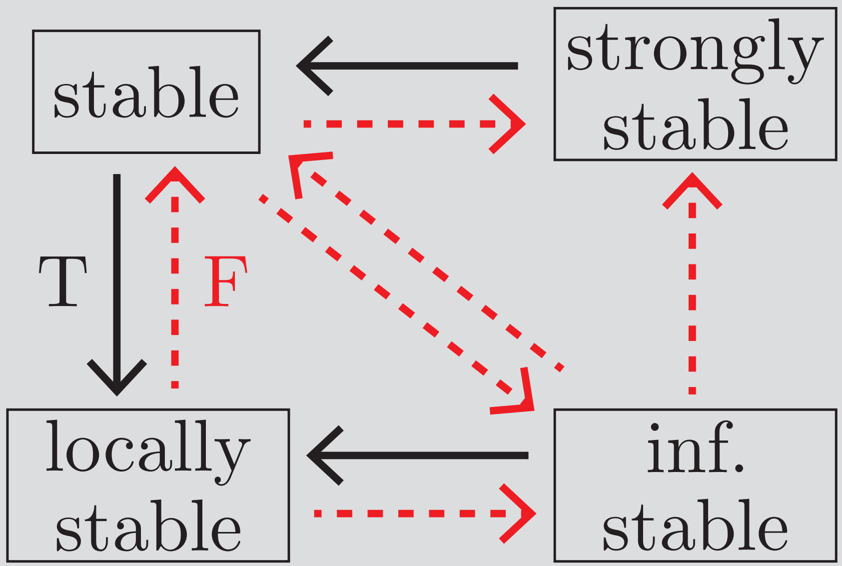
- f : strongly stable $\Rightarrow$ f : quasi-proper (du Plessis-Vosegaard)

  f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $f(C_f)$ s.t.

  $$f|_{f^{-1}(V)} : f^{-1}(V) \to V : \text{proper}$$

  e.g. $\exp(x)\sin x$ & $x^2 - y^2$: NOT quasi-proper

- Using the results we have explained, we can show:



◇ Remarks on problem 2 (3/3)

Problem 2

How are the four stabilities related for non-proper functions?

In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- $f \in C^{\infty}(N, P)$ is strongly **and** infinitesimally stable if and only if f is locally stable, quasi-proper and $f(C_f)$ is closed (du-Plessis-Vosegaard)

  Still, we have no reasonable condition implying only strong stability...

◇ Motivating problems (Summary)

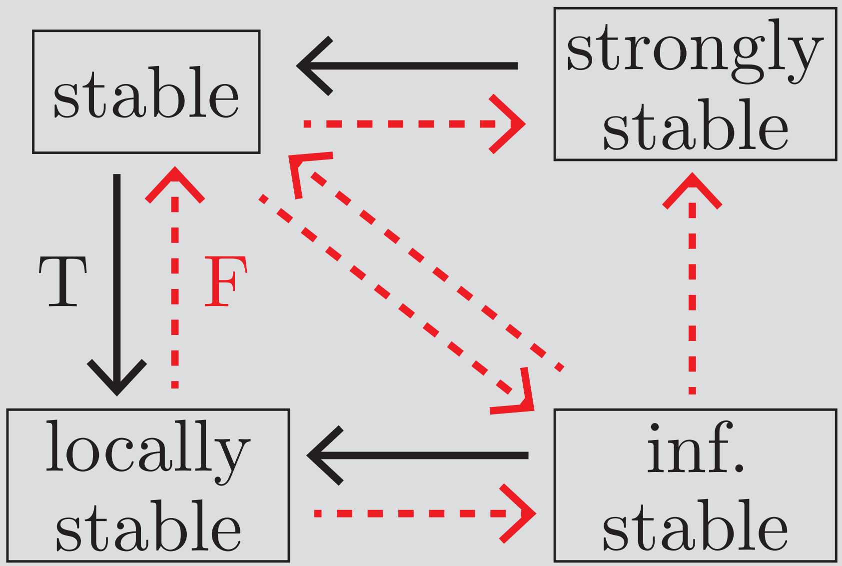
1. **detecting** (strong) **stability of non-proper functions.**

   e.g. Is $f(x,y) = x^2 - y^2$ stable?

   Note that f : **NOT** quasi-proper (thus **NOT** strongly stable).

2. **strongly stable** $\Rightarrow$ **infinitesimally stable?**

   The other implications are known to be True/False as follows:



§.2 Main result

Theorem (H.)

$f \in C^{\infty}(N, \mathbb{R})$: Morse function.

$\tau(f) := \{y \in \mathbb{R} \mid f$: "end-trivial" at $y\}$.

(the definition of end-triviality will be given soon...)

1. $f(C_f) \subset \tau(f) \Rightarrow f$: stable.
2. f : strongly stable $\Leftrightarrow$ f : quasi-proper

f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $f(C_f)$ s.t.

$f|_{f^{-1}(V)} : f^{-1}(V) \to V$: proper

◇ Remarks on the main result

- As we explained, f : strongly stable $\Rightarrow$ f : quasi-proper for $f \in C^\infty(N, P)$ (du Plessis-Vosegaard)
  We indeed show the converse of it for the case $P = \mathbb{R}$.

- Dimca's condition ($f(C_f) \cap (\mathcal{S}(f) \cup L(f)) = \emptyset$) is equivalent to ours ($f(C_f) \subset \tau(f)$). Indeed, $\tau(f) = \mathbb{R} \setminus (\mathcal{S}(f) \cup L(f))$ for $f \in C^\infty(\mathbb{R}, \mathbb{R})$, where

$$L(f) = \left\{ y \in \mathbb{R} \,\middle|\, y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\},$$

$$\mathcal{S}(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \,\middle|\, \{x_i\} : \begin{array}{l} \text{sequence in } C_f \text{ without} \\ \text{accumulation points} \end{array} \right\}.$$

◇ End-triviality

$V \subset N$: **neighborhood of the end**:$\Leftrightarrow$ $N \setminus V$: compact

Definition $f \in C^\infty(N, P)$, $y \in P$.

f is **end-trivial** at y if $\exists W \subset P$: neighborhood of y,

$\exists V \subset N$: open neighborhood of the end s.t.

- $f^{-1}(y) \cap V$ contains no critical points of f,
- $\exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V$: diffeomorphism

  s.t. $f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W$: projection

Roughly, end-triviality at y implies that f is the projection

"around the end of f^{-1}(nbhd. of y)".

$\exists W \subset P$: nbhd. of y, $\exists V \subset N$: open nbhd. of the end s.t.

- $f^{-1}(y) \cap V$ contains no critical points of f,
- $\exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V$: diffeomorphism
  s.t. $f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W$: projection

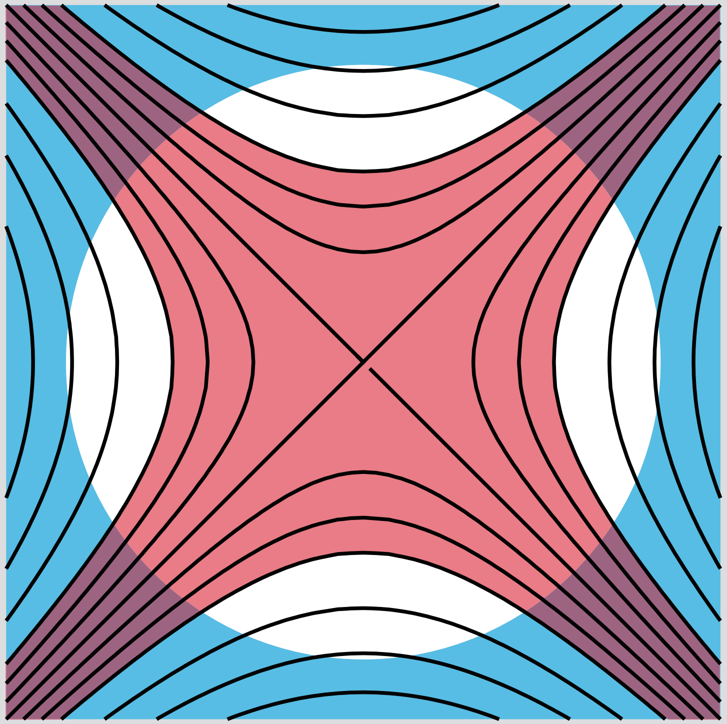
Example The fig. is contours of $f(x, y) := x^2 - y^2$ in $\mathbb{R}^2$.

Blue : outside of (sufficiently large) disk (which is V)

Red : preimage of nbhd. of $0 \in \mathbb{R}$ (which is $f^{-1}(W)$ for $y = 0$)

One can regard $f = p_2$ on Blue∩Red.
(i.e. $\exists \Phi$ with the desired property)

Thus, f is end-trivial at $0 \in \mathbb{R}$.



$\diamondsuit$ Main result (Again)

Theorem (H.)

$f \in C^\infty(N, \mathbb{R})$: Morse function.

$\tau(f) := \{y \in \mathbb{R} \mid f$: end-trivial at $y\}$.

1. $f(C_f) \subset \tau(f) \Rightarrow f$: stable.
2. f : strongly stable $\Leftrightarrow f$: quasi-proper

f : **quasi-proper** $:\Leftrightarrow \exists V \subset P$: neighborhood of $f(C_f)$ s.t.

$$f|_{f^{-1}(V)} : f^{-1}(V) \to V : \text{proper}$$

§.3 Applications

◇ detecting stability

Example $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $f(x, y) = x^2 - y^2$.

$C_f = \{0\}$ and $0 \in \tau(f)$ (as we checked) $\Rightarrow f$ is stable.

In general, end-triviality of semi-algebraic mappings has been studied in detail.

Definition $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$: semi-algebraic, $y \in \mathbb{R}$.

f satisfies the **Malgrange condition** at y

$:\Leftrightarrow \exists \delta > 0$, $\exists \varepsilon > 0$, $\exists V \subset \mathbb{R}^n$: nbhd. of the end s.t.

$$\|x\| \cdot \|\nabla f(x)\| > \varepsilon \text{ for any } x \in f^{-1}(y - \delta, y + \delta) \cap V.$$

Here, ∇f is the gradient of f.

Theorem (Folklore?) $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$: semi-algebraic.

If f satisfies the Malgrange condition at $y \in \mathbb{R}$, then f is end-trivial at y .

Corollary 1 (H.) $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$: Morse & semi-algebraic.

f is stable if it satisfies the Malgrange condition at $\forall y \in f(C_f)$.

Corollary 2 (H.) $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$: semi-algebraic.

$\exists \Sigma \subset \mathbb{R}^n$: Lebesgue measure zero set s.t.

$\forall a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \Sigma$, the function

$$f_a(x_1, \ldots, x_n) = f(x_1, \ldots, f_n) + \sum_{i=1}^{n} a_i x_i$$

is stable.

$\diamondsuit$ strong & infinitesimal stability

Corollary 3 (H.)

The function $f(x) = \exp(-x^2)\sin x$ is strongly stable but NOT infinitesimally stable.

We indeed show that f : Morse function, quasi-proper

& $f|_{C_f}$: NOT proper.

($f \in C^\infty(N, \mathbb{R})$: inf. stable $\Leftrightarrow$ f : Morse & $f|_{C_f}$: proper (Mather))

◇ Related topics (1/2)

- A sufficient condition for **topological** strong stability (for general N & P) is given by Murolo, du Plessis and Trotman.
- du Plessis-Vosegaard studied stability under another topology τV^{∞} of $C^{\infty}(N, P)$ (which is stronger than the Whitney topology). They indeed showed:

Theorem (du Plessis-Vosegaard)
Under the topology τV^{∞}, for a quasi-proper mapping, strong stability, stability, "quasi-infinitesimal stability" and local stability are all equivalent.

◇ Related topics (2/2)

- Little is known about stability for $\dim P > 1$.

  For example, the following problem is still open.

  Problem : Is there a non-proper stable mapping in $C^\infty(\mathbb{R}, \mathbb{R}^2)$? (w.r.t. the Whitney topology)

  Indeed, even the following simple (but non-proper) embedding is not stable!! (du Plessis-Vosegaard):

  $$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (\exp(x), 0).$$

  Note that f is quasi-proper, locally stable (in particular strongly stable w.r.t. τV^∞).

◇ Summary (what we gave)

- A sufficient condition for (strong) stability of $f \in C^\infty(N, \mathbb{R})$.

- The answers to the following questions:

1. Is $f(x, y) = x^2 - y^2$ stable? **Yes!**

2. strongly stable $\Rightarrow$ infinitesimally stable? **No!**

Thank you for your attention!!