Stability of non-proper functions

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\[ f : \mathbb{N} \to P : \text{proper} \iff \forall K \subset P : \text{compact}, \ f^{-1}(K) : \text{compact} \]

A function is a \( C^\infty \)-mapping to \( \mathbb{R} \) (i.e. \( P = \mathbb{R} \)).

**Assume that mfd's are \( C^\infty \), second countable & have no \( \partial \).**
§.1 Introduction

♦ Notations

- $N, P$ : manifolds

\[ C^\infty(N, P) := \{ f : N \to P : C^\infty\text{-mapping} \} \]

We endow $C^\infty(N, P)$ with the “Whitney $C^\infty$-topology”

(Roughly speaking, two mappings are close to each other under this topology iff they have close differentials.)

- $\text{Diff}(N) \subset C^\infty(N, N)$ : set of self-diffeomorphisms
  
  $\text{Diff}(N)$ is endowed with the relative topology

- For $f \in C^\infty(N, P)$, $C_f := \{ x \in N \mid \text{rank}(df_x) < \text{dim } P \}$.
Definition of stability

Definition

- \( f, g \in C^\infty(N, P) \) are right-left equivalent (\( f \sim g \))
  \[ \iff \exists \Phi \in \text{Diff}(N), \exists \phi \in \text{Diff}(P) \text{ s.t. } g = \phi \circ f \circ \Phi. \]

- \( f \in C^\infty(N, P) \) is stable (w.r.t. the Whitney topology)
  \[ \iff \exists \mathcal{U} \subset C^\infty(N, P) : \text{nbhd. of } f \text{ (w.r.t. the Whitney topology)} \]
  s.t. \( \forall g \in \mathcal{U} \) is right-left equivalent to \( f \).

It is in general difficult to check whether a given mapping is stable or not!!
\[ f : \text{stable} \iff \exists U \subset C^\infty(N, P) : \text{nbhd. of } f \text{ s.t. } g \sim f \text{ for } \forall g \in U. \]

\[ \Diamond \text{ Simple examples (1/2)} \]

\[ f_n \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ defined by } f_n(x) := x^n. \]

\textbf{Claim 1.} \( f_1 = \text{id}_\mathbb{R} \) is stable.

\textbf{Proof :} Define \( U := \left\{ g \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \forall t \in \mathbb{R}, \ g'(t) > \frac{1}{2} \right\}. \) Then,

\begin{itemize}
  \item \( U \) is an open nbhd. of \( f_1. \)
  \item By the inverse func., intermediate val. & mean val. theorems, \( U \subset \text{Diff}(\mathbb{R}), \) in particular \( f_1 \sim g \ (g = g \circ f_1 \circ \text{id}_\mathbb{R}) \) for \( \forall g \in U. \)
\end{itemize}

Thus, \( f_1 \) is stable. \( \square \)
$f$ : stable $: \iff \exists U \subset C^\infty(N, P) : \text{nbhd. of } f \text{ s.t. } g \sim f \text{ for } \forall g \in U.$

◊ Simple examples (2/2)

Claim 2. $f_3$ (in general $f_n$ for $n \geq 3$) is not stable.

Proof (for $n = 3$): Define $g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ by $g_t(x) := x^3 + \rho(x)tx$, where $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$, $\rho(x) \equiv 1$ for $|x| \leq 1$, $\rho(x) \equiv 0$ for $|x| \geq 2$.

• For $0 < \forall t \ll 1$, $g_t$ has no critical points, in particular $g_t \not\sim f_3$.

• $\mathbb{R} \ni t \mapsto g_t \in C^\infty(\mathbb{R}, \mathbb{R})$ is continuous, and thus
  \[ \forall U \subset C^\infty(\mathbb{R}, \mathbb{R}) : \text{open nbhd. of } f_3, \exists t > 0 \text{ s.t. } g_t \in U. \]

Thus, $f_3$ is not stable. □

How about $f_2$...? It is not so easy to show that it is stable...
Definition of infinitesimal stability

$\Gamma(E)$: set of sections of a vector bundle $E$.

**Definition**

$f \in C^\infty(N, P)$ is *infinitesimally stable*

\[ \iff \Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP)), \]

where

$df_* : \Gamma(TN) \to \Gamma(f^*TP)$ is defined by $df_*(\xi) := df \circ \xi$.

$f^* : \Gamma(TP) \to \Gamma(f^*TP)$ is defined by $f^*(\eta) := \eta \circ f$.

**Remark** (Motivation for infinitesimal stability)

$L_f : \text{Diff}(N) \times \text{Diff}(P) \to C^\infty(N, P), L_f(\Phi, \phi) := \phi \circ f \circ \Phi^{-1}$.

- stability $\iff$ image of $L_f$ contains a nbhd. of $f$.
- inf. stability $\iff$ the “differential $(dL_f)(\text{id}_N, \text{id}_P)$” is surjective.
\[ f: \text{inf. stable } \iff \Gamma(f^*TP) = df_*(\Gamma(TN)) + f^*(\Gamma(TP)). \]

\[ \diamond \text{ Simple examples (again) } \]

\[ f_n \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ defined by } f_n(x) := x^n. \]

**Claim 3.** \( f_2 \) is infinitesimally stable.

**Proof:** We can identify \( \Gamma(T\mathbb{R}) = \Gamma(f_2^*T\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R}). \)

Under these identifications, \( (df_2)_*(\xi) = 2x\xi \) and \( f_2^*(\xi) = \xi(x^2). \)

Since \( \xi(x) = \xi(0) + \int_0^1 \frac{d}{dt}(\xi(tx)) \, dt = \xi(0) + x \int_0^1 \frac{d\xi}{dt}(tx) \, dt \)

for \( \xi \in C^\infty(\mathbb{R}, \mathbb{R}), \ \Gamma(f_2^*T\mathbb{R}) = (df_2)_*(\Gamma(T\mathbb{R})) + f_2^*(\Gamma(T\mathbb{R})). \)
Stability for proper mappings (1/2)

Theorem (Mather 1970)

For \( f \in C^\infty(N, P) \): proper mapping, stability, infinitesimal stability, strong stability and “local stability” are all equivalent.

Definition  \( f \in C^\infty(N, P) \): strongly stable

\[ \iff \exists U \subset C^\infty(N, P) : \text{neighborhood of } f \]

\[ \exists (\Theta, \theta) : U \to \text{Diff}(N) \times \text{Diff}(P) : \text{continuous map} \]

\[ \text{s.t. } \forall g \in U, \theta(g) \circ g \circ \Theta(g) = f. \]
Stability for proper mappings (2/2)

We will only give several properties of “local stability”.

• local stability is the weakest condition of the four stabilities.
  i.e. (inf.) stable ⇒ locally stable for general (possibly non-proper) \( f \).

• In general, it is (relatively) easy to check local stability (Mather).
  e.g. \( f : \mathbb{N} \to \mathbb{R} \) (possibly non-proper) function is locally stable
  \( \iff f : \text{Morse function, that is,} \)
  \[ - \forall x \in C_f, \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (x) \right)_{i,j} \neq 0. \]
  \[ - f|_{C_f} : \text{injective.} \]

Thus, it is easy to check stability of proper mappings!!
Motivating problem 1

Problem 1
How can we detect (strong) stability of non-proper functions? e.g. Is $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by $f(x, y) = x^2 - y^2$ stable? Note that $f$ is infinitesimally stable but NOT strongly stable!! (will be seen later)
Remarks on problem 1 (1/2)

Problem 1
How can we detect (strong) stability of non-proper functions?
e.g. \( f(x, y) = x^2 - y^2 \): stable?

- \( f \): inf. stable \( \iff \) \( f \): loc. stable & \( f|_{C_f} \): proper (Mather).

In particular, infinitesimal stability is easily checked.
(since it is easy to check local stability.)

However, it is in general difficult to check (strong) stability!
Remarks on problem 1 (2/2)

Problem 1

How can we detect (strong) stability of non-proper functions?
e.g. \( f(x, y) = x^2 - y^2 \): stable?

- (Dimca) \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \): stable
  \[ \iff f : \text{locally stable} \land f(C_f) \cap (S(f) \cup L(f)) = \emptyset, \]
  where

  \[ L(f) = \left\{ y \in \mathbb{R} \mid y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\} \]

  \[ S(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \mid \{x_i\} : \text{sequence in } C_f \text{ without accumulation points} \right\} \]

  Thus, it is (somewhat) easy to check stability of \( f \in C^\infty(\mathbb{R}, \mathbb{R}). \)
\( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) : stable \( \iff \) \( f \) : locally stable \& \( f(C_f) \cap (S(f) \cup L(f)) = \emptyset \).

\[
L(f) = \left\{ \lim_{x \to \pm \infty} f(x) \right\}, \quad S(f) = \left\{ \lim_{i \to \infty} f(x_i) \left| \{x_i\} : \text{seq. in } C_f \text{ w/o accumulation pt's} \right. \right\}
\]

**Example** \( f : \mathbb{R} \to \mathbb{R}, \ f(x) := \exp(x) \sin x \).

Since \( f^{(k)}(x) = \frac{2^k}{2} \exp(x) \sin \left(x + \frac{k\pi}{4}\right) \), it is easy to see:

- \( C_f = \left\{ \frac{(4n + 3)\pi}{4} \in \mathbb{R} \left| n \in \mathbb{Z} \right. \right\}, \)
- \( f \) : Morse func. (i.e. \( f|_{C_f} \) : inj. \& \( \forall x \in C_f, \ f^{(2)}(x) \neq 0 \)).

Furthermore, \( S(f) = L(f) = \{0\} \) \& \( 0 \not\in f(C_f) \Rightarrow f \) : stable

On the other hand, \( (f|_{C_f})^{-1}([-1, 1]) \) : infinite discrete set \( \Rightarrow f \) : NOT infinitesimally stable (\( \because f|_{C_f} \) : not proper).
Motivating problem 2

Problem 2
How are the four stabilities related for non-proper functions? In particular, strongly stable $\Rightarrow$ infinitesimally stable?
Remarks on problem 2 (1/3)

Problem 2
How are the four stabilities related for non-proper functions?
In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- $f$ : strongly stable $\Rightarrow f$ : stable (obvious).
- $f$ : stable $\Rightarrow f$ : locally stable (Mather).
- $f$ : inf. stable $\Leftrightarrow f$ : loc. stable & $f|_{C_f}$: proper (Mather).
 Remarks on problem 2 (2/3)

• \( f \) : strongly stable \( \Rightarrow \) \( f \) : quasi-proper (du Plessis-Vosegaard)

\[ f : \text{quasi-proper} \iff \exists V \subset P : \text{neighborhood of } f(C_f) \text{ s.t.} \]
\[ f|_{f^{-1}(V)} : f^{-1}(V) \to V : \text{proper} \]

e.g. \( \exp(x) \sin x \) & \( x^2 - y^2 \): NOT quasi-proper

• Using the results we have explained, we can show:

\[
\begin{array}{ccc}
\text{stable} & \xrightarrow{F} & \text{strongly stable} \\
\text{T} & \xleftarrow{F} & \text{locally stable} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{stable} & \xrightarrow{F} & \text{strongly stable} \\
\text{T} & \xleftarrow{F} & \text{locally stable} \\
\end{array}
\]
Problem 2

How are the four stabilities related for non-proper functions? In particular, strongly stable $\Rightarrow$ infinitesimally stable?

- $f \in C^\infty(N, P)$ is strongly and infinitesimally stable if and only if $f$ is locally stable, quasi-proper and $f(C_f)$ is closed (du-Plessis-Vosegaard)

Still, we have no reasonable condition implying only strong stability...
Motivating problems (Summary)

1. detecting (strong) stability of non-proper functions.
   e.g. Is \( f(x, y) = x^2 - y^2 \) stable?
   Note that \( f \) : NOT quasi-proper (thus NOT strongly stable).

2. strongly stable \( \Rightarrow \) infinitesimally stable?
   The other implications are known to be True/False as follows:
## §.2 Main result

**Theorem (H.)**

\( f \in C^\infty(N, \mathbb{R}) \) : Morse function.

\( \tau(f) := \{ y \in \mathbb{R} \mid f \) : “end-trivial” at \( y \}\} \).

(the definition of end-triviality will be given soon...)

1. \( f(C_f) \subset \tau(f) \Rightarrow f \) : stable.

2. \( f \) : strongly stable \( \iff f \) : quasi-proper

\( f \) : quasi-proper \( \iff \exists V \subset P \) : neighborhood of \( f(C_f) \) s.t.

\[
\left. f \right|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V : proper
\]
 Remarks on the main result

- As we explained, $f : \text{strongly stable} \Rightarrow f : \text{quasi-proper}$ for $f \in C^\infty(N, P)$ (du Plessis-Vosegaard)
We indeed show the converse of it for the case $P = \mathbb{R}$.

- Dimca’s condition $(f(C_f) \cap (S(f) \cup L(f)) = \emptyset)$ is equivalent to ours $(f(C_f) \subset \tau(f))$. Indeed,
\[ \tau(f) = \mathbb{R} \setminus (S(f) \cup L(f)) \text{ for } f \in C^\infty(\mathbb{R}, \mathbb{R}), \]
where
\[ L(f) = \left\{ y \in \mathbb{R} \mid y = \lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \right\}, \]
\[ S(f) = \left\{ \lim_{i \to \infty} f(x_i) \in \mathbb{R} \mid \{x_i\} : \text{sequence in } C_f \text{ without accumulation points} \right\}. \]
\* \textbf{End-triviality} \\
\[ V \subset N : \text{neighborhood of the end} \iff N \setminus V : \text{compact} \]

\begin{center}
\begin{tabular}{|l|}
\hline
\textbf{Definition} \quad f \in C^\infty(N, P), \ y \in P. \\
\hline
\end{tabular}
\end{center}

\[ f \] is \textbf{end-trivial} at \( y \) if \( \exists W \subset P : \text{neighborhood of} \ y, \]
\( \exists V \subset N : \text{open neighborhood of the end} \) s.t.
\begin{itemize}
  \item \( f^{-1}(y) \cap V \) contains no critical points of \( f \),
  \item \( \exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V : \text{diffeomorphism} \)
\end{itemize}
\( \text{s.t. } f \circ \Phi = p_2 : (f^{-1}(y) \cap V) \times W \to W : \text{projection} \)

Roughly, end-triviality at \( y \) implies that \( f \) is the projection \( \text{“around the end of } f^{-1}(\text{nbhd. of } y)\text{”} \).
\[ \exists W \subset P: \text{nbhd. of } y, \exists V \subset N: \text{open nbhd. of the end s.t.} \]

- \[ f^{-1}(y) \cap V \text{ contains no critical points of } f, \]
- \[ \exists \Phi : (f^{-1}(y) \cap V) \times W \to f^{-1}(W) \cap V : \text{diffeomorphism} \]
  \begin{align*}
  \text{s.t. } f \circ \Phi &= p_2 : (f^{-1}(y) \cap V) \times W \to W : \text{projection}
  \end{align*}

**Example**  The fig. is contours of \[ f(x, y) := x^2 - y^2 \text{ in } \mathbb{R}^2. \]

**Blue**: outside of (sufficiently large) disk (which is \( V \))

**Red**: preimage of nbhd. of \( 0 \in \mathbb{R} \) (which is \( f^{-1}(W) \) for \( y = 0 \))

One can regard \( f = p_2 \) on \( \text{Blue} \cap \text{Red} \).

(i.e. \( \exists \Phi \) with the desired property)

Thus, \( f \) is end-trivial at \( 0 \in \mathbb{R} \).
Main result (Again)

Theorem (H.)

\( f \in C^\infty(N, \mathbb{R}) \) : Morse function.

\( \tau(f) := \{ y \in \mathbb{R} \mid f : \text{end-trivial at } y \} \).

1. \( f(C_f) \subset \tau(f) \Rightarrow f : \text{stable} \).

2. \( f : \text{strongly stable} \iff f : \text{quasi-proper} \)

\( f : \text{quasi-proper} \iff \exists V \subset P : \text{neighborhood of } f(C_f) \text{ s.t.} \)

\[ f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V : \text{proper} \]
§.3 Applications

◊ detecting stability

Example  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $f(x, y) = x^2 - y^2$.

$C_f = \{0\}$ and $0 \in \tau(f)$ (as we checked) $\Rightarrow f$ is stable.

In general, end-triviality of semi-algebraic mappings has been studied in detail.

**Definition**  $f \in C^\infty(\mathbb{R}^n, \mathbb{R}):$ semi-algebraic, $y \in \mathbb{R}$.

$f$ satisfies the **Malgrange condition** at $y$

$\iff \exists \delta > 0, \exists \varepsilon > 0, \exists V \subset \mathbb{R}^n :$ nbhd. of the end s.t.

$$\|x\| \cdot \|\nabla f(x)\| > \varepsilon \text{ for any } x \in f^{-1}(y - \delta, y + \delta) \cap V.$$  

Here, $\nabla f$ is the gradient of $f$. 
Theorem (Folklore?) $f \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$: semi-algebraic. If $f$ satisfies the Malgrange condition at $y \in \mathbb{R}$, then $f$ is end-trivial at $y$.

Corollary 1 (H.) $f \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$: Morse & semi-algebraic. $f$ is stable if it satisfies the Malgrange condition at $\forall y \in f(C_f)$.

Corollary 2 (H.) $f \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$: semi-algebraic. $\exists \Sigma \subset \mathbb{R}^n$: Lebesgue measure zero set s.t. $\forall a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \Sigma$, the function

$$f_a(x_1, \ldots, x_n) = f(x_1, \ldots, f_n) + \sum_{i=1}^{n} a_i x_i$$

is stable.
diamond strong & infinitesimal stability

Corollary 3 (H.)

The function $f(x) = \exp(-x^2) \sin x$ is strongly stable but NOT infinitesimally stable.

We indeed show that $f: \text{Morse function, quasi-proper}$

& $f|_{C_f}: \text{NOT proper}$.

($f \in C^\infty(N, \mathbb{R}) : \text{inf. stable } \Leftrightarrow f: \text{Morse} \& f|_{C_f}: \text{proper (Mather)})$
A sufficient condition for topological strong stability (for general $N$ & $P$) is given by Murolo, du Plessis and Trotman. du Plessis-Vosegaard studied stability under another topology $\tau V^\infty$ of $C^\infty(N, P)$ (which is stronger than the Whitney topology). They indeed showed:

**Theorem (du Plessis-Vosegaard)**

Under the topology $\tau V^\infty$, for a quasi-proper mapping, strong stability, stability, "quasi-infinitesimal stability" and local stability are all equivalent.
Related topics (2/2)

- Little is known about stability for $\dim P > 1$.

For example, the following problem is still open.

**Problem**: Is there a non-proper stable mapping in $C^\infty(\mathbb{R}, \mathbb{R}^2)$? (w.r.t. the Whitney topology)

Indeed, even the following simple (but non-proper) embedding is not stable!! (du Plessis-Vosegaard):

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(x) = (\exp(x), 0).$$

Note that $f$ is quasi-proper, locally stable (in particular strongly stable w.r.t. $\tau V^\infty$).
*Summary (what we gave)*

- A sufficient condition for (strong) stability of $f \in C^\infty(N, \mathbb{R})$.
- The answers to the following questions:
  1. Is $f(x, y) = x^2 - y^2$ stable? **Yes!**
  2. strongly stable $\Rightarrow$ infinitesimally stable? **No!**

Thank you for your attention!!