Stable map complexity and hyperbolic volumes of 3-manifolds

Masaharu Ishikawa

Keio University

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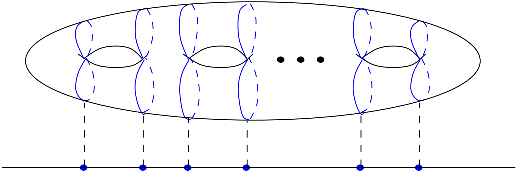
joint work with Yuya Koda (Hiroshima University)

Surface case (1/2)

M: a closed surface, $\chi(M) < 0$
$f : M \to \mathrm{R}$: the projection in the figure:

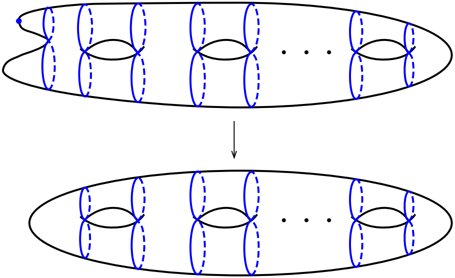


$c(f)$: the number of singular fibers of index 1.
The above map satisfies $c(f) = 2g$ and therefore we have

$$\mathrm{vol}(M) = 2\pi|\chi(M)| = 4(g-1)\pi = 2(c(f)-2)\pi$$

Surface case (2/2)

In general case, minimizing the number of singular fibers, we obtain the projection in the previous slide.



Idea of our study

$$\mathrm{vol}(M) = \min\{2(c(f)-2)\pi \mid f : M \to \mathbb{R} \text{ is a Morse function}\}$$

Main Result

Theorem (I.-Koda, 2017)

Let M be a closed, hyperbolic 3-manifold. If $\ell_{\min} > 2\pi$ then we have the following inequalities:

$$2\mathrm{smc}(M)V_{\mathrm{oct}}\left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \leq \mathrm{vol}(M) \leq 2\mathrm{smc}(M)V_{\mathrm{oct}}.$$

$V_{\mathrm{oct}} = 3.66\ldots$: **the hyperbolic volume of ideal regular octahedron**

$\mathrm{smc}(M)$: **the minimal number of certain singular fibers of a stable map $f : M \to \mathbb{R}^2$**

$\ell_{\min}$: **some positive real number**

Contents

Definition

Let M and N be smooth manifolds. Two smooth maps $f : M \to N$ and $g : M \to N$ are said to be **right-left equivalent** if there exist diffeomorphisms $\phi : M \to M$ and $\psi : N \to N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ f\downarrow & & \downarrow g \\ N & \xrightarrow{\psi} & N \end{array}$$

commutes.

Definition

A smooth map $f : M \to N$ is called a **stable map** if it is right-left equivalent to any smooth map g in a neighborhood of f in the space of smooth maps (with Whitney C^∞ topology).

Stable map (2/3)

EXAMPLE: **A Morse function $f : M \to \mathbb{R}$ whose critical points have distinct critical values is a stable map.**

Theorem (well-known)

Let M be a smooth 3-manifold and $f : M \to \mathbb{R}^2$ be a stable map to $\mathbb{R}^2$. Then f is locally given in one of the following forms:

(1) $(u, x, y) \mapsto (u, x)$

(2) $(u, x, y) \mapsto (u, x^2 + y^2)$ ... **definite fold**

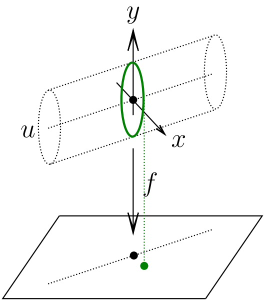
(3) $(u, x, y) \mapsto (u, x^2 - y^2)$ ... **indefinite fold**

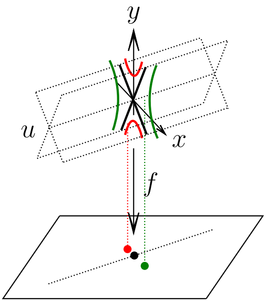
(4) $(u, x, y) \mapsto (u, y^2 + ux - x^3)$ ... **cusp**

Stable map (3/3)

(2) $(u, x, y) \mapsto (u, x^2 + y^2)$ ... **definite fold**

(3) $(u, x, y) \mapsto (u, x^2 - y^2)$ ... **indefinite fold**

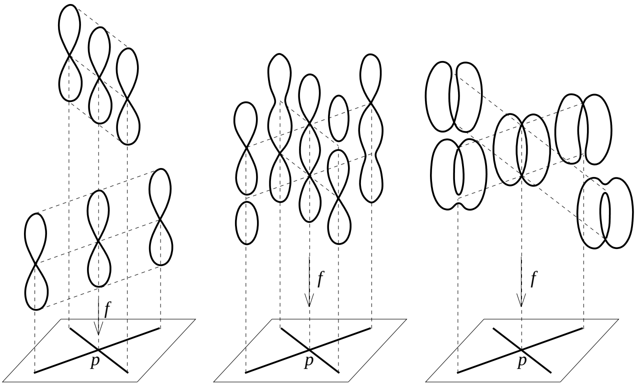




Theorem (Levin, 1965)

All cusps can be eliminated by a deformation of a stable map.

Singular fibers over a double point of $f(\mathrm{Sing}(f))$



Definition

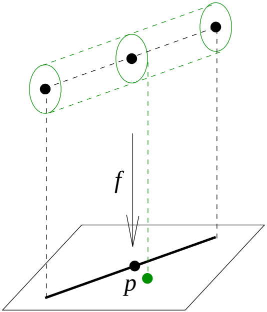
The singular fiber in the middle is called of type II^2 and the one on the right is of type II^3.

Define a stable map for the pair (M, L)

Definition

Let M be a closed, orientable 3-manifold and L be a link in M. A map $f : (M, L) \to \mathrm{R}^2$ is called a stable map of (M, L) if

- **$f : M \to \mathrm{R}^2$ is a stable map and**
- **L is contained in the set of definite folds of f.**



Definition of smc(M, L)

$\mathrm{II}^2(f)$: the set of singular fibers of f of type II^2
$\mathrm{II}^3(f)$: the set of singular fibers of f of type II^3
$|A|$: the number of elements in the set A

Definition

(1) The complexity $c(f)$ of a stable map $f : (M, L) \to \mathrm{R}^2$ is defined by

$$c(f) = |\mathrm{II}^2(f)| + 2|\mathrm{II}^3(f)|.$$

(2) smc$(M, L) = \min\{c(f) \mid f : (M, L) \to \mathrm{R}^2$ is a stable map$\}$ is called the stable map complexity of (M, L). We denote it by smc(M) if $L = \emptyset$.

Recall the main result

Main result (recall)

Let M be a closed, hyperbolic 3-manifold. If $\ell_{\min} > 2\pi$ then we have the following inequalities:

$$2\mathrm{smc}(M)V_{\mathrm{oct}}\left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^{2}\right)^{3/2} \le \mathrm{vol}(M) \le 2\mathrm{smc}(M)V_{\mathrm{oct}}.$$

Theorem (Saeki, 1996).

Let M be a closed, orientable 3-manifold. Then, smc$(M, L) = 0$ if and only if (M, L) is a graph manifold with a graph link L.

§2. Stein factorization

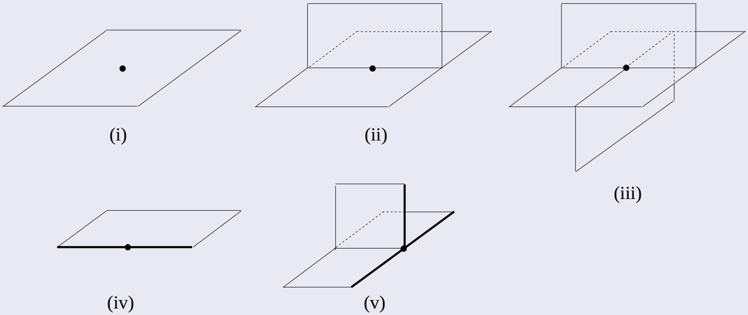
— and Turaev's reconstruction of M —

There is a notion of shadow in low-dimensional topology, which is a polyhedron almost same as a Stein factorization. Turaev's reconstruction is a method to reconstruct 3 (and 4)-manifolds from that polyhedron.

Simple polyhedron

Definition

A polyhedron P is called a simple polyhedron if a neighborhood of each point in P is homeomorphic to one of the following five models:



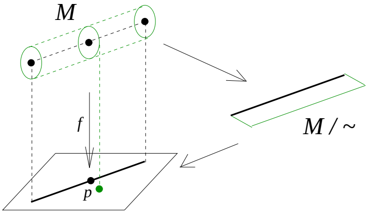
In this talk, we assume that the boundary ∂P consists of a disjoint union of circles.

Def.

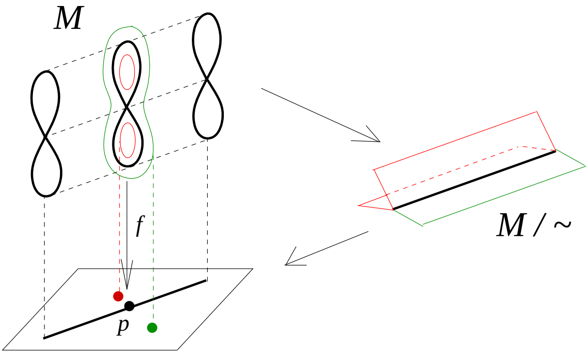
Let $f : M \to \mathrm{R}^2$ be a stable map. Define an equivalence relation $\sim$ between two points $x, y \in M$ by

$$x \sim y \Leftrightarrow f(x) = f(y) \text{ and } x \text{ and } y \text{ lie on the same component of } f^{-1}(f(x)).$$

The quotient space $M/\sim$ is called the **Stein factorization** of f.

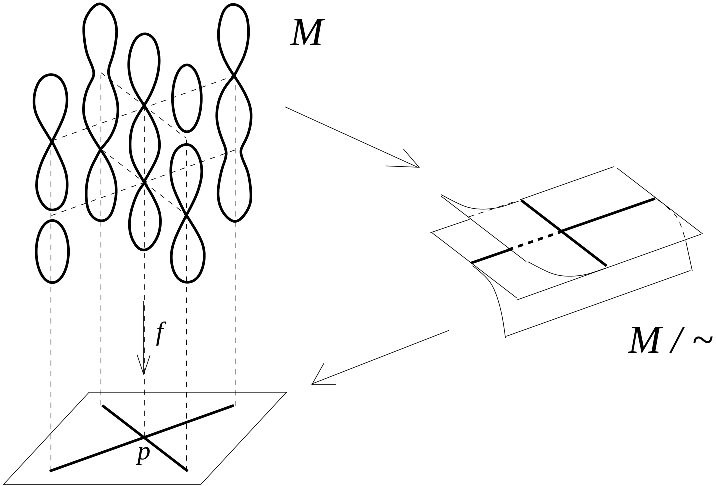


Indefinite fold



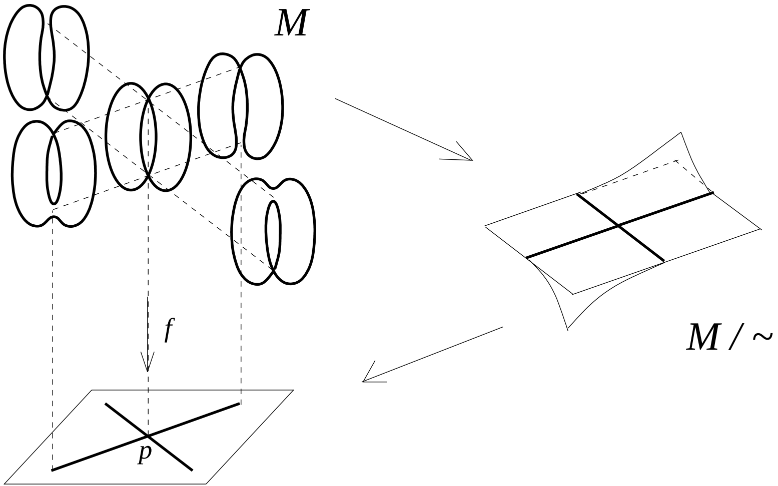
Stein factorization (3/4)

Singular fiber of type II^2

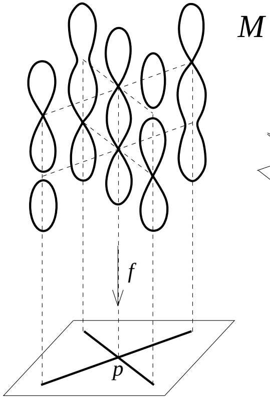


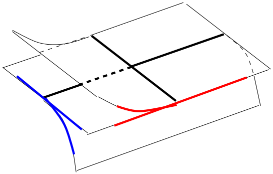
Stein factorization (4/4)

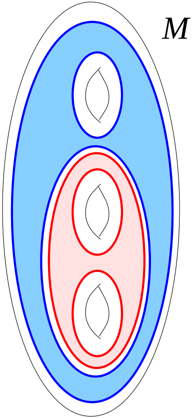
Singular fiber of type II^3

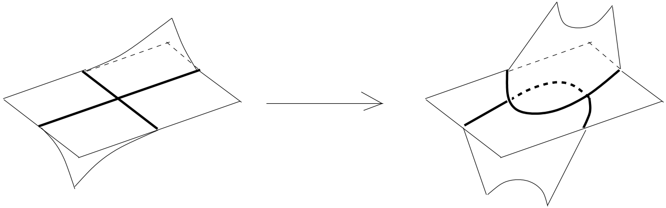


The neighborhood of a singular fiber of type II^2 in M









Deform as above so that the modified polyhedron is simple.

Definition (recall)

The complexity $c(f)$ of a stable map $f : (M, L) \to \mathrm{R}^2$ is defined by

$$c(f) = |\mathrm{II}^2(f)| + 2|\mathrm{II}^3(f)|.$$

Reconstruction (3/3)

The manifold M can be reconstructed from the Stein factorization with additional information assigned to each region, called a gleam: Let P be the Stein factorization of $f : M \to \mathrm{R}^2$.

(1) **Modify each vertex of type II^3 in P into two vertices of type II^2. Denote it by P'.**

(2) **Prepare a genus 3 handlebody for each vertex of P' and glue them according to the combinatorics of the singular set of P'. Let M' denote the obtained 3-manifold with boundary.**

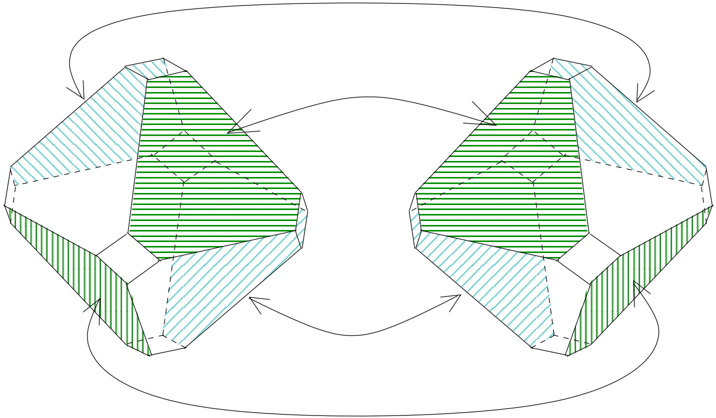
Note: The piece of M corresponding to each region of P' is a circle bundle over the region. The gleam is a kind of the euler number of this bundle. ℓ_{min} is determined by this information.

(3) **For each region R of P', glue $R \times S^1$ to M' according to the gleam, and then obtain M.**

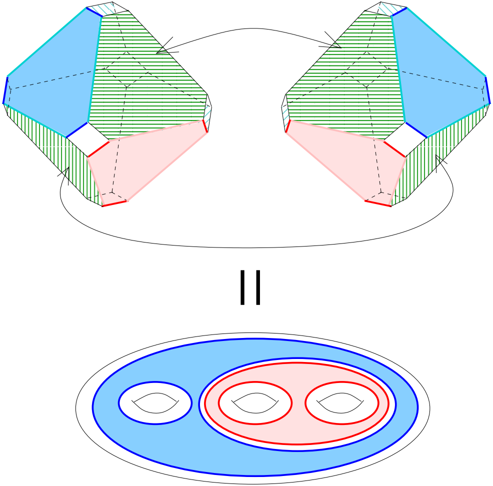
§3. Glue two regular ideal octahedra

[CT] F.Costantino and D.Thurston, *3-manifolds efficiently bound 4-manifolds*, J. Topol. **1** (2008), 703–745.

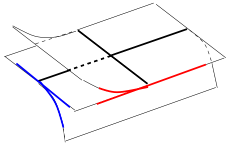
2 regular ideal octahedra (1/3)

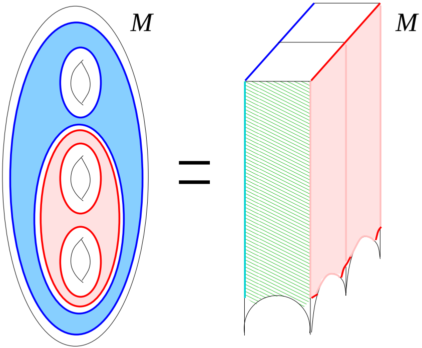


2 regular ideal octahedra (2/3)



2 regular ideal octahedra (3/3)





Lemma

Let $f : M \to \mathrm{R}^2$ be a stable map with Stein factorization P. The piece of M corresponding to a vertex of P has the hyperbolic volume $2V_{\text{oct}}$. Therefore, $\mathrm{vol}(M') = 2c(f)V_{\text{oct}}$, where M' is the union of these pieces.

§4 Results

[IK] M.Ishikawa and Y.Koda, *Stable maps and branched shadows of 3-manifolds, Math. Ann. 367 (2017), 1819-1863.*

- **Reconstruction of a stable map from a polyhedron**
- **Explicit construction of stable maps for (S^3, L)**

Proof of Main Theorem

The claim of the main theorem is

$$\mathbf{2}\mathrm{smc}(M)V_{\mathrm{oct}}\left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \leq \mathrm{vol}(M) \leq \mathbf{2}\mathrm{smc}(M)V_{\mathrm{oct}}.$$

Upperbound:

Note: The argument is same as what [CT] did for shadow complexity.

Since M is hyperbolic, M is obtained from M' by gluing copies of $D^2 \times S^1$. Then, by Thurston's hyperbolic Dehn surgery theorem, we have $\mathrm{vol}(M) \leq \mathrm{vol}(M') = 2c(f)V_{\mathrm{oct}}$.

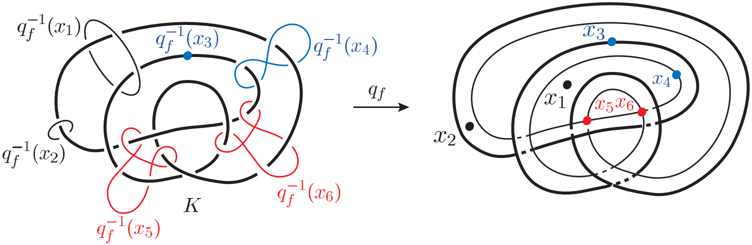
Lowerbound:

We use an inequality of Futer-Kalfagianni-Purcell. □

Theorem (I.-Koda).

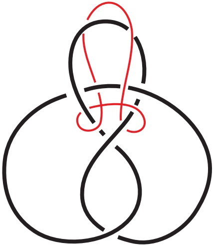
For each link in S^3, there exists a stable map of $f : (S^3, L) \to \mathbb{R}^2$ satisfying the following conditions:

- **the Stein factorization is contractible**
- **$|\mathrm{II}^2(f)| \leq$ the crossing number of L minus 2.**
- **$|\mathrm{II}^3(f)| = 0$ and no cusp.**



Example

Let $L = 4_1$ be the figure eight knot in S^3. The pair $(S^3, 4_1)$ admits a stable map f with $c(f) = 1$. Since 4_1 is hyperbolic, $\mathrm{smc}(S^3, 4_1) > 0$. Therefore, we have $\mathrm{smc}(S^3, 4_1) = 1$.



Closing remarks

- **Gromov generalized the upperbound to the higher-dimensional case $f : M^n \to N^{n-1}$.** *[M.Gromov, Singularities, expanders and topology of maps. I. Homology versus volume in the spaces of cycles, Geom. Funct. Anal.* **19** *(2009), 743–841.]*

- **Recently, Furutani and Koda gave a complete characterization of the hyperbolic links in the 3-sphere that admit stable maps into the real plane with exactly one singular fiber of type II^3. Such links are obtained from the link complements consisting of 10 ideal regular tetrahedra by Dehn filling.** *[R.Furutani, Y.Koda, Stable maps and hyperbolic links, arXiv:2103.00894 [math.GT]]*

Thank you for your attention.