Boundary maps from finitely generated groups to CAT(0) spaces

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Main Theorem

Γ: a finitely generated group with a random walk μ
Y: a locally compact CAT(0) space

Main Theorem (I.)

$\rho\colon \Gamma \to \mathrm{Isom}(Y)$: a homomorphism
If $\rho(\Gamma)$ does not fix a point in ∂Y, then either

- $\exists F \subset Y$ a flat subspace with $\rho(\Gamma)(F) = F$, or
- $\exists \varphi\colon \partial_P\Gamma \to \partial Y$: a canonical ρ-equivariant map.

Here $\partial_P\Gamma$ is the Poisson boundary of (Γ, μ) and ∂Y is the geometric boundary of Y. ρ-equivariant means that $\rho(\gamma)\varphi(\xi) = \varphi(\gamma\xi)$ for $\gamma \in \Gamma$, $\xi \in \partial_P\Gamma$.

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The existence of such an equivariant boundary map often implies a strong rigidity result on group actions.

CAT(0) spaces

$Y = (Y, d)$: a complete metric space
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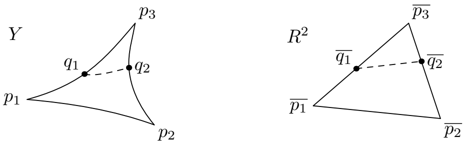

$d_Y(p_i, p_j) = d_{R^2}(\overline{p_i}, \overline{p_j})$, $d_Y(q_1, q_2) \le d_{R^2}(\overline{q_1}, \overline{q_2})$

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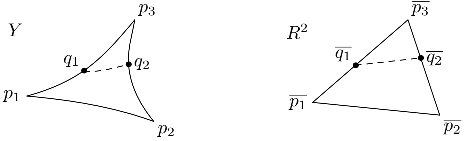

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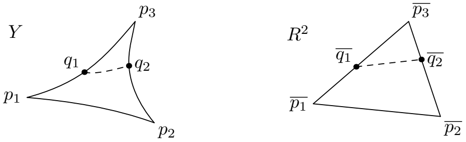

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Ex 2. Trees, Euclidean buildings are CAT(0) spaces.

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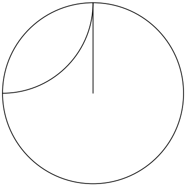

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

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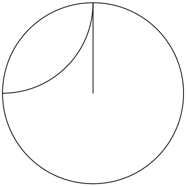

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Ex 3. If Y is a tree, ∂Y is a Cantor set.

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Note. Every isometry of Y extends to a homeo of ∂Y.

Boundary maps in rigidity theory

Ex (Mostow Rigidity). If $\Gamma, \Lambda < \mathrm{Isom}(\mathbb{H}^n)$ discrete, $n \geq 3$, $\Gamma \overset{\text{isomorphic}}{\cong} \Lambda$ and $\mathbb{H}^n/\Gamma$ is compact. Then $\mathbb{H}^n/\Gamma \overset{\text{isometry}}{\cong} \mathbb{H}^n/\Lambda$.

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- f lifts to a ρ-equiv map $\tilde{f} \colon \mathbb{H}^n \to \mathbb{H}^n$ ($\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$).

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- $\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a conformal diffeomorphism, and recovers a ρ-equivariant isometry $f_0\colon \mathbb{H}^n \to \mathbb{H}^n$, which descends to an isometry $h\colon \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$.

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- $\bar{f}\colon \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a ρ-equivariant conformal diffeo, and recovers a ρ-equivariant isometry $f_0\colon \mathbb{H}^n \to \mathbb{H}^n$, which descends to an isometry $h\colon \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$.

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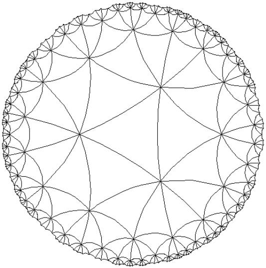

Since $\mathbb{H}^n/\Gamma$ is compact, $\Gamma\xi$ is dense in $\partial\mathbb{H}^n$ for any $\xi \in \partial\mathbb{H}^n$. Thus $\bar{f}$ is comletely determined by $\bar{f}(\xi)$ for some $\xi \in \partial\mathbb{H}^n$.

Random walk on a group and its Poisson boundary

Γ: a finitely generated group

Consider a random walk on Γ; $\mu\colon \Gamma \times \Gamma \to [0,1]$ with

- probability $\sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') = 1$,
- symmetric $\mu(\gamma, \gamma') = \mu(\gamma', \gamma)$,
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• $\mu(\gamma, \gamma')$ is the transition probability of $\gamma \to \gamma'$.

• $\mu^n(\gamma, \gamma')$ is the *n*-step transition probability of $\gamma \to \gamma'$ after *n* steps:

$$\mu^n(\gamma, \gamma') = \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \Gamma} \mu(\gamma, \gamma_1)\mu(\gamma_1, \gamma_2) \ldots \mu(\gamma_{n-1}, \gamma')$$

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• The Poisson boundary of Γ w.r.t. μ is the probability space describing the distribution of "∞-step random walk":

$$\partial_P \Gamma \text{ ``}=\text{''} \lim_{n \to \infty} (\Gamma, \mu^n(e, \cdot)).$$

If the random walk "diverges", then $\partial_P \Gamma$ can be viewed as a boundary at ∞ of Γ, and Γ acts on $\partial_P \Gamma$.

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This refines a theorem due to Bader-Duchesne-Lécuruex. Our φ is canonical: φ is obtained as an extension of an orbit map $\gamma \mapsto \rho(\gamma)p$ to $\partial_P \Gamma$ for $p \in Y$.

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Note 1. The map φ is not only measurable but Lipschitz continuous w.r.t. "Tits metric" on $\partial_P \Gamma$ and ∂Y.

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Note 2. Under certain probablistic assumption on $\rho(\Gamma)$, we can drop the local compactness assumption on Y.

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Note 3. If Y is hyperbolic in the sense of Gromov, we don't have to assume Y to be locally compact, and F becomes the image of a geodesic line.

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Corollary (Haettel)

Y: a CAT(0) space, hyperbolic in the sense of Gromov
$\rho\colon \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{Isom}(Y)$: a homomorphism, $n \geq 3$.
Then either

- $\rho(\Gamma)$ fixes a point in ∂Y, or
- $\rho(\Gamma)$ leaves a geodesic line invariant.

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The proof uses a ρ-equivariant harmonic map $f\colon \Gamma \to Y$.

- ρ-equivariance: $\rho(\gamma) f(\gamma') = f(\gamma\gamma')$ for $\forall \gamma, \gamma' \in \Gamma$,
- harmonicity: f minimizes μ-energy E_μ

$$E_\mu\colon f \mapsto \int_{\gamma \in \Gamma} d(f(e), f(\gamma))^2 d\mu(e, \gamma)$$

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Let $(\Omega, \mathbb{P}) = (\Gamma, \mu) \times \cdots \times (\Gamma, \mu) \times \ldots$ and set $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, $\gamma_n(\omega) = \omega_1 \ldots \omega_n \in \Gamma$.
By Kingman's subadditive ergodic theorem,

$$\frac{d(f(e), f(\gamma_n(\omega)))}{n} \to \exists C \in [0, \infty) \quad \text{in } L^1(\Omega, \mathbb{P}).$$

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If $C > 0$, then by Karlsson-Margulis, $f(\gamma_n(\omega)) \to \exists \xi \in \partial Y$ a.a. $\omega \in (\Omega, \mathbb{P})$.
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Since f is μ-harmonic,

$$u\colon Y \to \mathbb{R} \text{ convex } \Rightarrow \Delta_\mu f^* u \leq 0,$$

where

$$\Delta_\mu h(\gamma) = h(\gamma) - \int_{\gamma' \in \Gamma} h(\gamma\gamma') d\mu(\gamma, \gamma').$$

If $C = 0$, $\exists \xi \in Y \cup \partial Y$ such that $\Delta f^* u \equiv 0$, where $u(p) = d(\xi, p)$ or $u(p) = b_\xi(p)$, which means that u has the weakest possible convexity, which leads us to see the convex hull of $f(\Gamma)$ is flat.