Boundary maps from finitely generated groups to CAT(0) spaces

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Main Theorem

Γ: a finitely generated group with a random walk μ
Y: a locally compact CAT(0) space

Main Theorem (I.)

ρ: Γ → Isom(Y): a homomorphism
If ρ(Γ) does not fix a point in ∂Y, then either
- ∃F ⊂ Y a flat subspace with ρ(Γ)(F) = F, or
- ∃φ: ∂PΓ → ∂Y: a canonical ρ-equivariant map.

Here ∂PΓ is the Poisson boundary of (Γ, μ) and ∂Y is the geometric boundary of Y. ρ-equivariant means that ρ(γ)φ(ξ) = φ(γξ) for γ ∈ Γ, ξ ∈ ∂PΓ.
Main Theorem

\( \Gamma \): a finitely generated group with a random walk \( \mu \)
\( Y \): a locally compact CAT(0) space

Main Theorem (I.)

\( \rho : \Gamma \to \text{Isom}(Y) \): a homomorphism

If \( \rho(\Gamma) \) does not fix a point in \( \partial Y \), then either
- \( \exists F \subset Y \) a flat subspace with \( \rho(\Gamma)(F) = F \), or
- \( \exists \varphi : \partial_\rho \Gamma \to \partial Y \): a canonical \( \rho \)-equivariant map.

Here \( \partial_\rho \Gamma \) is the Poisson boundary of \((\Gamma, \mu)\) and \( \partial Y \) is the geometric boundary of \( Y \). \( \rho \)-equivariant means that
\( \rho(\gamma)\varphi(\xi) = \varphi(\gamma \xi) \) for \( \gamma \in \Gamma \), \( \xi \in \partial_\rho \Gamma \).

The existence of such an equivariant boundary map often implies a strong rigidity result on group actions.
CAT(0) spaces

\( Y = (Y, d) \): a complete metric space

- \( c: [0, T] \rightarrow Y \) is a \textbf{geodesic} if \( \forall t, t' \in [0, T] \), \( d(c(t), c(t')) = |t - t'| \).
CAT(0) spaces

$Y = (Y, d)$: a complete metric space

• $c: [0, T] \rightarrow Y$ is a geodesic if $\forall t, t' \in [0, T]$, $d(c(t), c(t')) = |t - t'|$.

• $Y$ is a geodesic space if $\forall p, q \in Y$, $\exists$ a geodesic $c$ joining $p$ and $q$. 

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Ex 1. A simply connected Riemannian manifold with nonpositive sectional curvature is CAT(0) space.

Ex 2. Trees, Euclidean buildings are CAT(0) spaces.
CAT(0) spaces

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- \( c: [0, T] \rightarrow Y \) is a geodesic if \( \forall t, t' \in [0, T], \)
  \[ d(c(t), c(t')) = |t - t'|. \]
- \( Y \) is a geodesic space if \( \forall p, q \in Y, \exists \) a geodesic \( c \) joining \( p \) and \( q \).
- \( Y \) is a CAT(0) space if every geodesic triangle is thinner than Euclidean one.

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- \( Y \) is a **CAT(0) space** if every geodesic triangle is thinner than Euclidean one.

\[
\begin{align*}
  d_Y(p_i, p_j) &= d_{R^2}(\overline{p_i}, \overline{p_j}), \\
  d_Y(q_1, q_2) &\leq d_{R^2}(\overline{q_1}, \overline{q_2})
\end{align*}
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**Ex 1.** A simply connected Riemannian manifold with nonpositive sectional curvature is \( \text{CAT}(0) \) space.

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Boundary of CAT(0) spaces

- $c : [0, \infty) \to Y$ is a geodesic ray if $\forall t, t' \in [0, \infty)$, \[ d(c(t), c(t')) = |t - t'|. \]
Boundary of CAT(0) spaces

• $c : [0, \infty) \to Y$ is a geodesic ray if $\forall t, t' \in [0, \infty), \ d(c(t), c(t')) = |t - t'|$.
• Geodesic rays $c, c'$ are asymptotic ($c \sim c'$) if $\exists M > 0$ s.t. $d(c(t), c'(t)) \leq M$ for $\forall t \in [0, \infty)$. 

Note. Every isometry of $Y$ extends to a homeo of $\partial Y$. 
Boundary of CAT(0) spaces

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- The boundary of $Y$ is defined by $\partial Y = \{\text{geodesic rays}\}/\sim$. 
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**Ex 1.** In $\mathbb{R}^n$, $c \sim c'$ iff $c$ and $c'$ are parallel; $\partial \mathbb{R}^n = S^{n-1}$. 

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Ex 1. In $\mathbb{R}^n$, $c \sim c'$ iff $c$ and $c'$ are parallel; $\partial \mathbb{R}^n = S^{n-1}$.

Ex 2. For hyperbolic $n$-space $\mathbb{H}^n$, $\partial \mathbb{H}^n = S^{n-1}$:

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

$$g_{\mathbb{H}^n} = \frac{4}{(1 - |x|^2)^2} \sum_{i=1}^{n} dx_i^2$$
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**Ex 3.** If $Y$ is a tree, $\partial Y$ is a Cantor set.
Boundary of CAT(0) spaces

- Geodesic rays \( c, c' \) are asymptotic (\( c \sim c' \)) if \( \exists M > 0 \) s.t. \( d(c(t), c'(t)) \leq M \) for \( \forall t \in [0, \infty) \).
- The boundary of \( Y \) is defined by \( \partial Y = \{ \text{geodesic rays} \}/ \sim \).

Note. Every isometry of \( Y \) extends to a homeo of \( \partial Y \).
Ex (Mostow Rigidity). If $\Gamma, \Lambda < \text{Isom}(\mathbb{H}^n)$ discrete, $n \geq 3$, isomorphic $\Gamma \cong \Lambda$ and $\mathbb{H}^n/\Gamma$ is compact. Then $\mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda$. 
Boundary maps in rigidity theory

**Ex (Mostow Rigidity).** If $\Gamma, \Lambda < \text{Isom}(\mathbb{H}^n)$ discrete, $n \geq 3$, isomorphic $\Gamma \cong \Lambda$ and $\mathbb{H}^n/\Gamma$ is compact. Then $\mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda$.

- $\exists f : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$: a homotopy equivalence. $f$ induces an isomorphism $\rho : \Gamma \to \Lambda$. 

\begin{align*}
\text{Ex (Mostow Rigidity).} & \quad \text{If } \Gamma, \Lambda \subset \text{Isom}(\mathbb{H}^n) \text{ discrete, } n \geq 3, \\
& \quad \text{isomorphic } \Gamma \cong \Lambda \text{ and } \mathbb{H}^n/\Gamma \text{ is compact. Then } \mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda. \\
& \quad \exists f : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda: \text{ a homotopy equivalence. } f \text{ induces an isomorphism } \rho : \Gamma \to \Lambda.
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Boundary maps in rigidity theory

Ex (Mostow Rigidity). If \( \Gamma, \Lambda \subset \text{Isom}(\mathbb{H}^n) \) discrete, \( n \geq 3 \), \( \Gamma \cong \Lambda \) and \( \mathbb{H}^n/\Gamma \) is compact. Then \( \mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda \).

- \( \exists f : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda \): a homotopy equivalence. \( f \) induces an isomorphism \( \rho : \Gamma \to \Lambda \).
- \( f \) lifts to a \( \rho \)-equiv map \( \tilde{f} : \mathbb{H}^n \to \mathbb{H}^n \) \( (\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)) \).
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- $\exists f : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$: a homotopy equivalence. $f$ induces an isomorphism $\rho : \Gamma \to \Lambda$.
- $f$ lifts to a $\rho$-equiv map $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ ($\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$).
- By hyperbolicity of $\mathbb{H}^n$ and cocompactness of $\Gamma$, $\tilde{f}$ extends to a $\rho$-equivariant homeomorphism $\tilde{f} : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$. 
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**Ex (Mostow Rigidity).** If $\Gamma, \Lambda \leq \text{Isom}(\mathbb{H}^n)$ discrete, $n \geq 3$, isomorphic $\Gamma \cong \Lambda$ and $\mathbb{H}^n/\Gamma$ is compact. Then $\mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda$.

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- $f$ lifts to a $\rho$-equiv map $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ ($\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$).
- By hyperbolicity of $\mathbb{H}^n$ and cocompactness of $\Gamma$, $\tilde{f}$ extends to a $\rho$-equivariant homeomorphism $\bar{f} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.
- $\bar{f} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a conformal diffeomorphism, and recovers a $\rho$-equivariant isometry $f_0 : \mathbb{H}^n \to \mathbb{H}^n$, which descends to an isometry $h : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$. 

Since $\mathbb{H}^n/\Gamma$ is compact, $\Gamma\xi$ is dense in $\partial\mathbb{H}^n$ for any $\xi \in \partial\mathbb{H}^n$. Thus $\bar{f}$ is completely determined by $\bar{f}(\xi)$ for some $\xi \in \partial\mathbb{H}^n$. 
Ex (Mostow Rigidity). If $\Gamma, \Lambda < \text{Isom}(\mathbb{H}^n)$ discrete, $n \geq 3$, $\Gamma \cong \Lambda$ and $\mathbb{H}^n/\Gamma$ is compact. Then $\mathbb{H}^n/\Gamma \cong \mathbb{H}^n/\Lambda$.

\[ \bar{f}: \partial \mathbb{H}^n \to \partial \mathbb{H}^n \] is a $\rho$-equivariant conformal diffeo, and recovers a $\rho$-equivariant isometry $f_0: \mathbb{H}^n \to \mathbb{H}^n$, which descends to an isometry $h: \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Lambda$. 

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Random walk on a group and its Poisson boundary

\( \Gamma: \) a finitely generated group

Consider a **random walk** on \( \Gamma; \) \( \mu: \Gamma \times \Gamma \to [0, 1] \) with

- probability \( \sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') = 1, \)
- symmetric \( \mu(\gamma, \gamma') = \mu(\gamma', \gamma), \)
- \( \Gamma \)-invariant \( \mu(\gamma \gamma', \gamma \gamma'') = \mu(\gamma', \gamma''), \)
- irreducible \( \{ \gamma \in \Gamma \mid \mu(e, \gamma) \neq 0 \} \) generates \( \Gamma. \)
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\( \mu(\gamma, \gamma') \) is the transition probability of \( \gamma \to \gamma' \).
Random walk on a group and its Poisson boundary

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- probability $\sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') = 1$,
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- $\Gamma$-invariant $\mu(\gamma \gamma', \gamma \gamma'') = \mu(\gamma', \gamma'')$,
- irreducible $\{\gamma \in \Gamma \mid \mu(e, \gamma) \neq 0\}$ generates $\Gamma$.

- $\mu(\gamma, \gamma')$ is the transition probability of $\gamma \to \gamma'$.
- $\mu^n(\gamma, \gamma')$ is the $n$-step transition probability of $\gamma \to \gamma'$ after $n$ steps:

$$\mu^n(\gamma, \gamma') = \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \Gamma} \mu(\gamma, \gamma_1) \mu(\gamma_1, \gamma_2) \cdots \mu(\gamma_{n-1}, \gamma')$$
Random walk on a group and its Poisson boundary

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  \]
- The Poisson boundary of \( \Gamma \) w.r.t. \( \mu \) is the probability space describing the distribution of "\( \infty \)-step random walk":
  \[
  \partial_P \Gamma = = \lim_{n \to \infty} (\Gamma, \mu^n(e, \cdot)).
  \]

If the random walk "diverges", then \( \partial_P \Gamma \) can be viewed as a boundary at \( \infty \) of \( \Gamma \), and \( \Gamma \) acts on \( \partial_P \Gamma \).
Main Theorem and Proof

$\Gamma$: a finitely generated group with a random walk $\mu$
$Y$: a locally compact CAT(0) space

Main Theorem (I.)

$\rho: \Gamma \to \text{Isom}(Y)$: a homomorphism
If $\rho(\Gamma)$ does not fix a point in $\partial Y$, then either

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- \( \exists \varphi: \partial P \Gamma \to \partial Y: \text{a canonical } \rho\text{-equivariant map} \).

This refines a theorem due to Bader-Duchesne-Lécuruex. Our \( \varphi \) is canonical: \( \varphi \) is obtained as an extension of an orbit map \( \gamma \mapsto \rho(\gamma)p \) to \( \partial P \Gamma \) for \( p \in Y \).
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Note 1. The map φ is not only measurable but Lipschitz continuous w.r.t. “Tits metric” on ∂ₚΓ and ∂Y.
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Note 2. Under certain probabilistic assumption on \( \rho(\Gamma) \), we can drop the local compactness assumption on \( Y \).
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Note 3. If \( Y \) is hyperbolic in the sense of Gromov, we don’t have to assume \( Y \) to be locally compact, and \( F \) becomes the image of a geodesic line.
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Corollary (Haettel)

\( Y \): a CAT(0) space, hyperbolic in the sense of Gromov

\[ \rho : \text{SL}(n, \mathbb{Z}) \rightarrow \text{Isom}(Y) : \text{a homomorphism, } n \geq 3. \]

Then either

- \( \rho(\Gamma) \) fixes a point in \( \partial Y \), or
- \( \rho(\Gamma) \) leaves a geodesic line invariant.
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The proof uses a \( \rho \)-equivariant harmonic map \( f : \Gamma \rightarrow Y \).

- \( \rho \)-equivariance: \( \rho(\gamma)f(\gamma') = f(\gamma \gamma') \) for \( \forall \gamma, \gamma' \in \Gamma \),
- harmonicity: \( f \) minimizes \( \mu \)-energy \( E_\mu \)

\[ E_\mu : f \mapsto \int_{\gamma \in \Gamma} d(f(e), f(\gamma))^2 d\mu(e, \gamma) \]
Main Theorem and Proof

Let \((\Omega, \mathbb{P}) = (\Gamma, \mu) \times \cdots \times (\Gamma, \mu) \times \cdots\) and set
\[\omega = (\omega_1, \omega_2, \ldots) \in \Omega, \quad \gamma_n(\omega) = \omega_1 \ldots \omega_n \in \Gamma.\]

By Kingman's subadditive ergodic theorem,

\[
\frac{d(f(e), f(\gamma_n(\omega)))}{n} \to \exists C \in [0, \infty) \quad \text{in } L^1(\Omega, \mathbb{P}).
\]
Main Theorem and Proof

Let \((\Omega, \mathbb{P}) = (\Gamma, \mu) \times \cdots \times (\Gamma, \mu) \times \cdots\) and set \(\omega = (\omega_1, \omega_2, \ldots) \in \Omega, \ \gamma_n(\omega) = \omega_1 \ldots \omega_n \in \Gamma.\)

By Kingman’s subadditive ergodic theorem,

\[
\frac{d(f(e), f(\gamma_n(\omega))))}{n} \rightarrow \exists C \in [0, \infty) \quad \text{in} \ L^1(\Omega, \mathbb{P}).
\]

If \(C > 0,\) then by Karlsson-Margulis, \(f(\gamma_n(\omega)) \rightarrow \exists \xi \in \partial Y\)
a.a. \(\omega \in (\Omega, \mathbb{P}).\)
If \(C = 0,\) then using the harmonicity of \(f,\) we can show that the convex hull of \(f(\Gamma)\) is flat.
Main Theorem and Proof

If $C = 0$, then using the harmonicity of $f$, we can show that the convex hull of $f(\Gamma)$ is flat.

Since $f$ is $\mu$-harmonic,

$$u : Y \to \mathbb{R} \text{ convex} \Rightarrow \Delta_\mu f^* u \leq 0,$$

where

$$\Delta_\mu h(\gamma) = h(\gamma) - \int_{\gamma' \in \Gamma} h(\gamma \gamma') d\mu(\gamma, \gamma').$$

If $C = 0$, $\exists \xi \in Y \cup \partial Y$ such that $\Delta f^* u \equiv 0$, where $u(p) = d(\xi, p)$ or $u(p) = b_\xi(p)$, which means that $u$ has the weakest possible convexity, which leads us to see the convex hull of $f(\Gamma)$ is flat.