Recent Progress on the Strominger-Yau-Zaslow conjecture

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Outline of the Talk

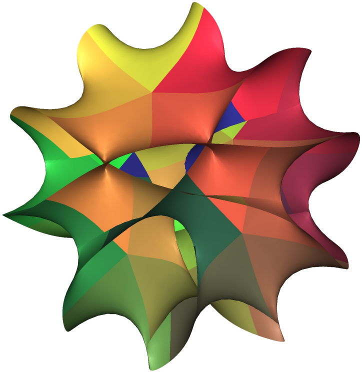
- SYZ Conjecture & Set-up of the Geometry
- SYZ Fibrations of Del Pezzo Surfaces and their Dual Fibrations
- Application: Discovery of New Ricci-Flat Metrics

The Easter Egg

Calabi-Yau Manifolds

A complex n-fold X is a Calabi-Yau manifold if it has

- holomorphic volume form Ω. $K_X := \wedge^n T^*X \cong \mathcal{O}_X$
- a Kähler form ω s.t. $\omega^n = c_n \Omega \wedge \bar{\Omega}$. Ricci-flat



Examples of Calabi-Yau Manifolds

- Elliptic curves, complex tori and K3 surfaces.
- degree 5 hypersurface in $\mathbb{P}^4$ (quintic 3-fold).
  This follows from Yau's proof of Calabi conjecture.
- (Tian-Yau '90)
  If Y is Fano & $D \in |-K_Y|$ smooth, then $X = Y \setminus D$ is CY.

Strominger-Yau-Zaslow Conjecture

Conjecture (Strominger-Yau-Zaslow '96)

- *Calabi-Yau manifolds admit* **special Lagrangian** *torus fibration near large complex structure limit.*
- *Mirror Calabi-Yau are constructed by dual torus fibration.*

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1. (Harvey-Lawson '82) A submanifold L in X is special Lagrangian if $\omega|_L = 0$, $\mathrm{Im}\Omega|_L = 0$.
2. (Duistermaat '80) Any compact fibre of a Lagrangian fibration is topologically a torus.

Why SYZ Conjecture?

The SYZ conjecture is important in various aspects:

- It gives a geometric description of Calabi-Yau manifolds.
- It provides a recipe to construct the mirror $\check{X}$.
- (Leung-Yau-Zaslow) It gives a guidance of how branes mirror to each other in the homological mirror symmetry.

Examples of Special Lagrangian Fibrations

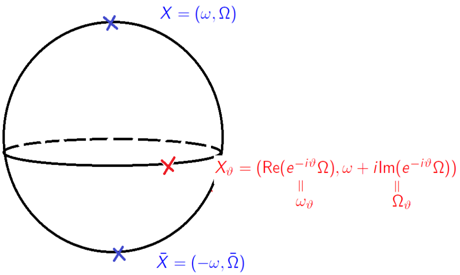
- $X = \mathbb{C}/\langle 1, \tau \rangle$ with $\omega = \frac{i}{2} dz \wedge d\bar{z}$, $\Omega = dz$, $\tau \in i\mathbb{R}_+$. Then

$$\begin{aligned} X &\to S^1 \\ z &\mapsto \mathrm{Im} z \end{aligned}$$

- (Harvey-Lawson '82) $X' \to B$ Calabi-Yau surface with elliptic fibration, then a suitable HK ration X admits a special Lagrangian fibration.

HyperKähler Geometry on K3 Surfaces

- Calabi-Yau surfaces are hyperKähler, i.e.
  - X admits a Kähler form ω.
  - (ω, Ω) induces an S^2-family of complex structures.



Then holomorphic curves in X $\Leftrightarrow$ special Lagrangians in X_ϑ.

Previous Results

- (Gross '99) Topological torus fibration on quintic 3-folds.
- (Gross '00) Examples for toric Calabi-Yau manifolds.
- (Symington-Leung '03) Almost toric symplectic 4-folds.
- (Joyce '03) Some local model with codimesion one discriminant locus.
- (Castano-Bernard, Matessi '06) Lagrangian fibrations on quintic.
- ......
- (Li '19) Weak SYZ conjecture on Fermat hypersurfaces.

New Special Lagrangian Fibrations

Theorem (Collins-Jacob-L. '19)

Y =Fano surface or RES, $D \in |-K_Y|$ smooth.
Then $X = Y \backslash D$ admits a special Lagrangian fibration with a special Lagrangian section with respect to the Tian-Yau metric.

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This solves conjectures of Yau and Auroux '07.

1. $Y = \mathbb{P}^2$, with 3 nodal singular fibres.
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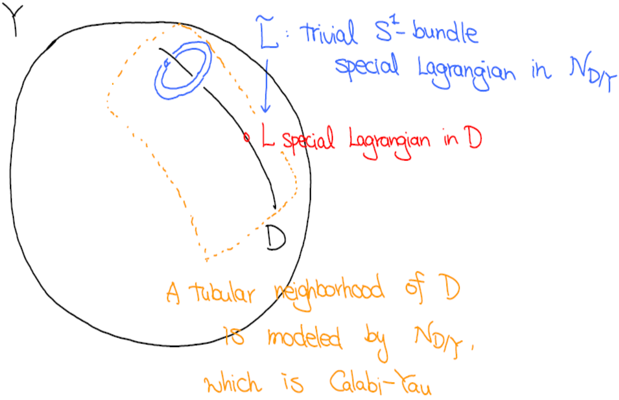
Theorem (Collins-Jacob-L. '20)

Dual Special Lagrangian Fibration *on the mirror of X.*

Mirrors can be connected by two HK rotations.

Step 1: Model Special Lagrangian Tori



Step 2: Lagrangian Mean Curvature Flow

- Let L be a graded Lagrangian submanifold in X, i.e., $\exists$ the phase $\theta : L \to \mathbb{R}$ is the function such that

$$\Omega|_L = e^{i\theta} vol_L.$$

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  L is a special Lagrangian if θ is a constant.
- The mean curvature $\vec{H} = J\nabla\theta$ and the mean curvature flow is given by evolving family of immersions $F_t : L \to X$ with

$$\frac{\partial}{\partial t} F_t = \vec{H}.$$

- (Smoczyk) Maslov zero Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds.
- Smooth Convergent Limit of LMCF gives Special Lagrangians.

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- $\check{X}$ HK rotation of X and L becomes an elliptic curve.
- Deformation of L in $\check{X}$ is open & closed.
  1. (McLean, Voisin) Deformation of smooth holo. Lagrangian is unobstructed.
  2. (closedness) Gromov-Sachs-Uhlenbech compactness theorem for the degenerate geometry.
  3. (openness) Analyze the possible singular fibres and their deformations.

Application to "New" Ricci-Flat Metrics

Theorem (Collins-Jacob-L.'20)

Given $(\check{Y}, \check{D})$, there exists an extra $\mathbb{R}$-family of Ricci-flat metrics on $\check{X}$ with Hein's metrics are indexed by $\mathbb{Z}$.

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- Take $\check{X}$ as in the 2nd theorem which is complete hyperKähler.
- If HK metric on $\check{X}$ is Hein's metric, then there exists a special Lagrangian fibration on $\check{X}$.

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- $D \cong \mathbb{C}/(\mathbb{Z} \oplus a\mathbb{Z}i), a \in \mathbb{R}_+$ and leads to a contradiction for general choice of D.

The Easter Egg

Fun application in algebraic geometry:

Theorem (? classical result)

Let $E \subseteq \mathbb{P}^2$ be a smooth cubic curve. Then there exists

$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \subseteq Aut(\mathbb{P}^2)$$

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1. The corresponding rational elliptic surface $\check{Y}$ is extremal $I_9 I_1^3$.
2. $\check{\Omega}$ is meromorphic.
3. $\exists \mathbb{Z}_3 \oplus \mathbb{Z}_3 \in \mathrm{Aut}(\check{Y})$ preserving $\check{\Omega}$ and $\check{\omega}$. Uniqueness theorem!
4. The corresponding automorphisms of X extend over E.

THANK YOU!