Simple Hurwitz groups and eta invariant

Takayuki Morifuji

(Keio Univ.)

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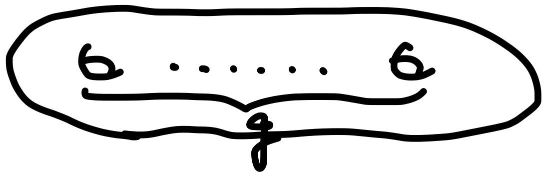
§1. Introduction



X : a compact Riemann surface of genus $g \geq 2$

$G := \mathrm{Aut}(X)$: the automorphism group of X

Fact (Hurwitz) $|G| \leq 84(g-1)$ (Schwartz : $|G| < \infty$)

If "$=$" holds, $\begin{cases} X \text{ is called Hurwitz surface.} \\ G \text{ is called Hurwitz group.} \end{cases}$

$\Leftrightarrow$ G is a nontrivial finite quotient of the triangle group $\Delta(2,3,7) = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle$

Ex.

G	$\|G\|$	X	g	
$PSL(2, \mathbb{F}_7)$	168	Klein surface	3	← the smallest Hurwitz gp
$PSL(2, \mathbb{F}_8)$	504	Macbeath surf.	7	
$PSL(2, \mathbb{F}_{13})$	1092	$\exists$ three non-isom. surfaces	14	

Ex. (Macbeath) $PSL(2, \mathbb{F}_q)$ is a Hurwitz group

$$\Longleftrightarrow \begin{cases} \text{(i)} & q = 7 \\ \text{(ii)} & q = p \equiv \pm 1 \pmod 7 \quad \text{or} \\ \text{(iii)} & q = p^3 \text{ where } p \equiv \pm 2 \text{ or } \pm 3 \pmod 7 \end{cases}$$

Rmk.
$$\begin{cases} \text{(i), (iii)} \Rightarrow \exists! \text{ Hurwitz surf. } X \text{ s.t. } \mathrm{Aut}(X) = PSL(2, \mathbb{F}_q) \\ \text{(ii)} \Rightarrow \exists \text{ three Hurwitz surfaces for each } q \end{cases}$$

$\Rightarrow$ $\exists$ ∞ly many Hurwitz groups!

Thm. 1 $\varphi \in PSL(2, \mathbb{F}_7)$ is reducible $\Longleftrightarrow$ $\eta(M_\varphi) = 0$

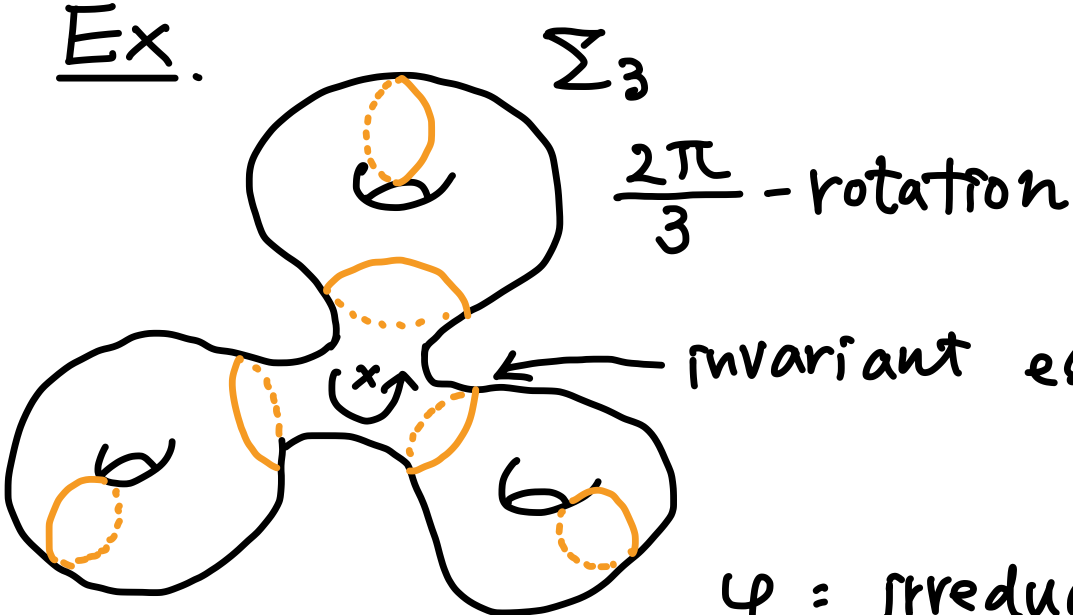
Nielsen - Thurston classification of $\varphi \in \mathrm{Diff}_+(\Sigma_g)$

(1) finite order

(2) reducible $\overset{def}{\Longleftrightarrow}$ φ leaves some essential 1-submanifold of Σ_g invariant.

(3) pseudo-Anosov

Ex.



$(1) \cap (2) \neq \phi$

φ : irreducible $\overset{def}{\Longleftrightarrow}$ φ is not reducible

$M_\varphi = \Sigma_g \times [0,1]/\sim$: a mapping torus (equipped with the metric induced from the product one)

$\eta(M_\varphi) = \frac{1}{3}\int_W P_1 - \mathrm{Sign}\, W$: the eta inv. of the signature op.

W : an ori. cpt. Riem. 4-mfd with the product metric near $\partial W = M_\varphi$

P_1 : the 1st Pontrjagin form of the metric

$\mathrm{Sign}\, W$: the signature of W

<u>Rmk</u> If $\partial W = \phi \Rightarrow \frac{1}{3}\int_W P_1 = \mathrm{Sign}\, W$ "Hirzebruch signature formula"

<u>Thm</u> 2. $G = PSL(2, \mathbb{F}_q)$ as in Ex(Macbeath)

If $q > 7 \Rightarrow \forall \varphi \in G$ reducible & $\eta(M_\varphi) = 0$

* $PSL(2, \mathbb{F}_7)$: a simple gp

What's the generalization of this viewpoint?

<u>Prop</u> 1. G : a simple Hurwitz group $\not\cong PSL(2, \mathbb{F}_7)$

$\Rightarrow \forall \varphi \in G$ reducible

<u>Rmk</u> $\exists \varphi \in PSL(2, \mathbb{F}_7)$ irreducible (of order 7)

<u>Que</u>. $\eta(M_\varphi) = 0$ $(\forall \varphi \in G)$ for G in Prop 1?

<u>Thm 3</u> Let G be one of the following simple Hurwitz groups (1) ~ (12). Then $\forall \varphi \in G$ reducible & $\eta(M_\varphi) = 0$.

Infinite family of simple groups

(1) all but 62 of the simple alternating groups A_n $(n \geq 5)$,

(2) the Chevalley gp. $G_2(q)$; $q = p^n \geq 5$ (p : prime) & $p \neq 3, 7$

Sporadic simple groups :

(3) the 1^{st} Janko gp J_1

(4) the Hall-Janko gp J_2

(5) the Rudvalis gp Ru

(6) the smallest Conway gp Co_3

(7) the Fisher gp Fi_{22}

(8) the Harada-Norton gp HN

(9) the Lyons gp Ly

(10) the Thompson gp Th

(11) the Fisher gp Fi_{24}'

(12) the Monster M

§2 Proof of Thm 3 (Sketch)

Prop 2 G : a Hurwitz gp s.t. all conjugacy classes of orders 3 & 7 are "real"

$\Longrightarrow$ $\eta(M_\varphi) = 0$ ($\forall \varphi \in G$).

A conjugacy class c of a finite group G is real if $\chi(c) \in \mathbb{R}$ holds for $\forall \chi \in \mathrm{Irr}(G)$.

$\mathrm{Irr}(G)$: the set of irreducible characters of G.

(1) Using the character formula of the symmetric group S_n, we can calculate irreducible characters of A_n

(2) We use Malle's result. $G_2(q)$: a Hurwitz gp $\Longleftrightarrow$ $q = p^n$ (p : prime) & $q \geq 5$

If $p \neq 3, 7$ $\Longrightarrow$ the conj. classes of orders 3 & 7 are rational (hence real).

(3) ~ (12) : We can check the assumption using character tables of finite simple groups (cf. ATLAS of Finite Groups).

§ 3. Proof of Props (We only show Prop 1).

Prop (Kasahara) $\varphi \in \mathcal{M}_g = \pi_0 \mathrm{Diff}_+ \Sigma_g$; of order m

the mapping class gp of Σ_g

(1) φ : irreducible $\Rightarrow$ $m \geq 2g+1$

(2) φ : reducible $\Rightarrow$ $m \leq 2g+2$, moreover if g : odd $\Rightarrow m \leq 2g$

Prop (Vdovin)

G : a nonabelian finite simple group s.t. $G \not\cong PSL(2, \mathbb{F}_q)$,

$q = p^k$ (p : prime)

A : an abelian subgroup of G

$\Rightarrow |A|^3 < |G|$

Prop 1 G : a simple Hurwitz gp $\not\cong PSL(2,\mathbb{F}_7) \Rightarrow \forall \varphi \in G$ reducible

X : a Hurwitz surface with $G = \mathrm{Aut}(X) \Rightarrow |G| = 84(g-1)$

m_0 : the maximal order of elements in G

$\underset{\text{Vdovin}}{\Longrightarrow}$ $m_0^3 < |G|$ if G : simple & $G \not\cong PSL(2,\mathbb{F}_8)$

If $m_0 < 2g+1 \overset{\text{Kasahara}}{\Longrightarrow} \forall \varphi \in G$ reducible

$$\begin{aligned} 2g+1-m_0 &= 2\left(\frac{1}{84}|G|+1\right)+1-m_0 \\ &= \frac{1}{42}|G| - m_0 + 3 \\ &> \frac{1}{42}|G| - |G|^{\frac{1}{3}} \end{aligned}$$

Since $G \not\cong PSL(2,\mathbb{F}_8)$, we have

$$|G| > |PSL(2,\mathbb{F}_{13})| = 1092 > 10^3$$

$$\therefore\ 2g+1-m_0 > \frac{1}{42}|G|^{\frac{1}{3}}\left(|G|^{\frac{2}{3}}-42\right) > \frac{10}{42}(10^2-42) > 0 \,/\!/$$

When $G = PSL(2, \mathbb{F}_q)$ as in Ex (Macbeath) (ii), (iii),

$$m_0 = \begin{cases} \dfrac{q+1}{\gcd(2, q-1)} & \text{if } q = p^k \ (k > 1) \\ q & \text{if } q = p > 2 \end{cases}$$

$$|PSL(2, \mathbb{F}_q)| = \begin{cases} \frac{1}{2} q(q^2 - 1) & q : \text{odd} \\ q(q^2 - 1) & q : \text{even} \end{cases}$$

Using these values, we can check $m_0 < 2q+1$ for $PSL(2, \mathbb{F}_q)$ //