q-deformations of Stability Conditions & Quadratic Differentials

with A. Ikeda

$\mathrm{QStab} = \mathrm{QQuad}$

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q-Deformations of categories, Stab & Quad

with. Akishi Ikeda.

§1. Motivations

- Homological Mirror Symmetry (Kontsevich)

$$\boxed{DFuk(X) \cong D^b(Coh X^\vee)}$$

classically Calabi-Yau-3

- A conj. geometric picture:

Bridgeland space of stability conditions $\boxed{\mathrm{Stab}\, D \cong \mathcal{M}_{cpt}(X^\vee)}$ Moduli space of complex str.

- Remark

In general: it is even hard to construct geo. Stab.

• Known results:

<u>Thm</u> (Bridgeland–Smith)

$\mathrm{DFuk}(X_{\mathbf{S}})$

$$\boxed{\mathrm{Stab}^\circ \mathcal{D}_3(\mathbf{S}) \cong \mathrm{FQuad}_3(\mathbf{S})}$$

King–Q's version

<u>Thm</u> (Haiden–Katzarkov–Kontsevich)

$$\boxed{\mathrm{Stab}^\circ \mathcal{D}_\infty(\mathbf{S}) \cong \mathrm{FQuad}_\infty(\mathbf{S})}$$

$:= \mathrm{TFuk}(\mathbf{S})$

Aim: use HKK to obtain a q-deformation of BS.

& with application to representation theory & cluster theory.

(i.e. relate $\mathcal{D}_3(\mathbf{S})$ & $\mathcal{D}_\infty(\mathbf{S})$ categorically)

§2. Categories' q-deformation.

- $Q = A_n$ quiver (ADE, acyclic, QP from surfaces & general)

$\mathcal{D}_\infty(Q) := D^b(kQ)$ ($\cong \mathrm{TFuk}(S)$ in HKK)

$\boxed{K\,\mathcal{D}_\infty(Q) \cong \mathbb{Z}^n}$, where $n = \# Q_0$.

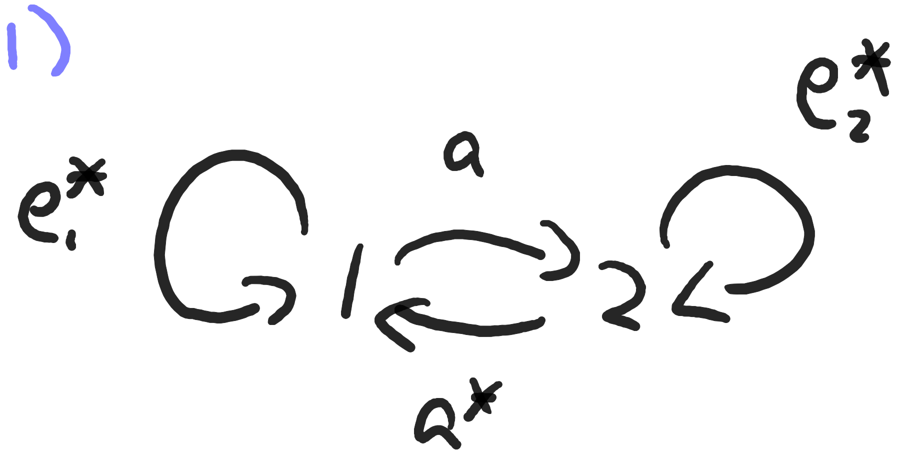
$\boxed{\mathbb{Z} \oplus \mathbb{Z}X}$ grading

Construction (IQ, after Ginzburg & Keller)

- $\overline{Q}$ = double of $Q = (\underbrace{Q_0, Q_1}, Q_1^*, Q_0^*)$

$\mathbb{Z}^2$-grading on $\overline{Q}$: $(0,0)$ $(2,-1)$ $(1,-1)$

diff. d with $\deg = (1,0)$



Consider dga $\Gamma_X Q := (k\overline{Q}, d)$

$$d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]$$

if $\exists W$, $\deg = (3,-1)$

$D_X(Q) := D_{fd}(\Gamma_X Q)$ which is Calabi-Yau-X $\boxed{[X]}$

for $X = (0,1)$-grading shift (Adams grading)

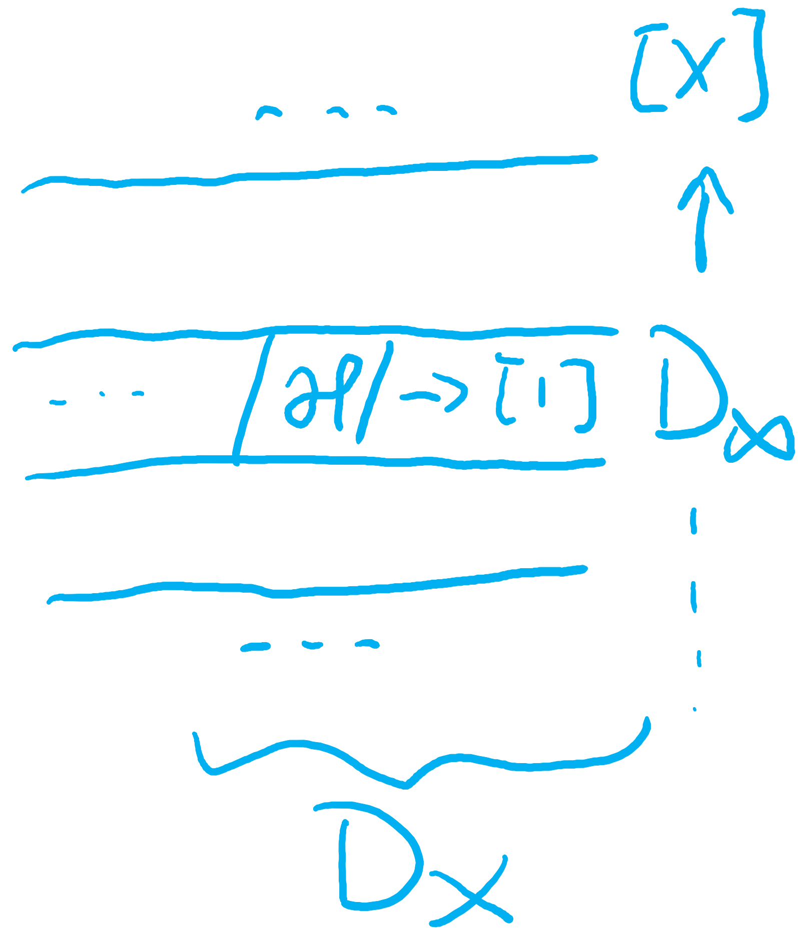
Key observation: $\boxed{K D_X(Q) = R^{\oplus n}}$ $R = \mathbb{Z}[q^{\pm 1}]$

Lem $\exists$ f.f. embedding $D_\infty(Q) \hookrightarrow D_X(Q)$ as an X-baric heart.

Recall that:

mod $kQ =: \mathcal{H}_Q \hookrightarrow D_\infty(Q)$

& $\langle \mathcal{H}_Q[\mathbb{Z}] \rangle = D_\infty(Q)$



Analogously: $\langle D_\infty(Q)[\mathbb{Z}X] \rangle = D_X(Q)$

§3. Cluster-X categories

$\Gamma_X Q \to \Gamma_3 Q$

$(0,1) = (3,0)$

$\Downarrow$

$[X] = [3]$

Classically (BMRRT, Keller)

$$\mathcal{C}_2(Q) := D_\infty(Q) / \tau_0[-1]$$

Amiot, Keller, Guo:

$$\boxed{0 \to \operatorname{per} \Gamma_3 Q \to D_{fd} \Gamma_3 Q \to \mathcal{C}_2(Q) \to 0}$$

IQ:

$$\uparrow /\!\!/ [X-3] \qquad \uparrow /\!\!/ [X-3] \qquad \uparrow /[X-3] \text{ “}=\text{” } \tau_0[-1]$$

$$\boxed{0 \to \operatorname{per} \Gamma_X Q \to D_X(Q) \to \mathcal{C}_X(Q) \to 0}$$

$$\mathcal{C}_X(Q) \cong D_\infty(X)$$

(this picture in full generality is in working progress)

§4. Stability condition's q-deformation

Def. $\mathcal{D}$ is a triangulated cat.

A stab. cond. $\sigma = (Z, \mathcal{P})$ consists of

$$\begin{cases} \text{a central charge } Z : K\mathcal{D} \to \mathbb{C}. \\ \text{a slicing } \mathcal{P} = \{\mathcal{P}(\phi) \mid \phi \in \mathbb{R}\} \end{cases}$$

$=$ a $\mathbb{R}$-collection of t-structures.

& they are compactible.

Thm (B): $\boxed{\text{Stab}\,\mathcal{D} \text{ forms a } \mathbb{C}\text{-mfd of } \dim = n}$

for $n = \text{rank}\, K\mathcal{D}$ $(<\infty)$

- $\exists$ 2 natural actions:

$$s \in \mathbb{C} \curvearrowright \operatorname{Stab} D \curvearrowleft \operatorname{Aut} \ni \Phi$$

$$\sigma = (Z, P) \in \operatorname{Stab} D$$

$$s \cdot (Z, P) = (e^{-i\pi s} \cdot Z, P_{\operatorname{Re}(s)})$$

$$\Phi(Z, P) = (Z \circ \Phi^{-1}, \Phi \circ P)$$

- $\exists$ local coordinate:

$$\mathcal{Z}: \boxed{\begin{aligned} \operatorname{Stab} D &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(KD, \mathbb{C}) \\ \sigma = (Z, P) &\longmapsto Z \end{aligned}}$$

(+ tech. condition)

IQ's q-deformation. $\boxed{q = e^{i\pi s}}$

$D = D_\infty \rightsquigarrow D_X$ with $X \in \mathrm{Aut}$ s.t.

$KD \cong \mathbb{Z}^n \qquad KD \cong R^{\oplus n}, \quad R = \mathbb{Z}[q^{\pm 1}].$

Def. A q-stab. cond. is a pair (σ, s) s.t.

$0°)$ σ is a Bridgeland stab. cond., $s \in \mathbb{C}$

$1°)$ $\boxed{s \cdot \sigma = X(\sigma)}$

$\Longleftrightarrow$ $1.1°$ $Z \in \mathrm{Hom}_R(KD_X, \mathbb{C}_s)$

$1.2°$ $P(\phi)[X] = P(\phi + \mathrm{Re}(s))$

$\mathbb{C}_s = \mathbb{C}$ is a R-mod

$q \cdot z = e^{i\pi s} \cdot z$

Thm (IQ) $\mathrm{QStab}_s \mathcal{D}_X$ is a $\mathbb{C}$-mfd with $\dim = n$.

<u>Specialization</u> to $s = N \in \mathbb{Z}_{\geq 2}$

e.g. quiver case

$$\mathcal{D}_N := \mathcal{D}_X / [X - N] \quad \text{CY-}N$$

<u>Thm</u> $\boxed{\mathrm{QStab}_N \mathcal{D}_X \hookrightarrow \mathrm{Stab}\, \mathcal{D}_N}$ open + closed.

§5. Quadratic differentials' q-deformation

S: a Riemann surface

A meromorphic quad. diff is a section of $\omega_S^{\otimes 2}$

Locally $\phi(z) = g(z)\, dz^{\otimes 2}$

$\Updownarrow$

a foliation = a direction in $\mathbb{P}(TS)$ for $\forall x \in S$

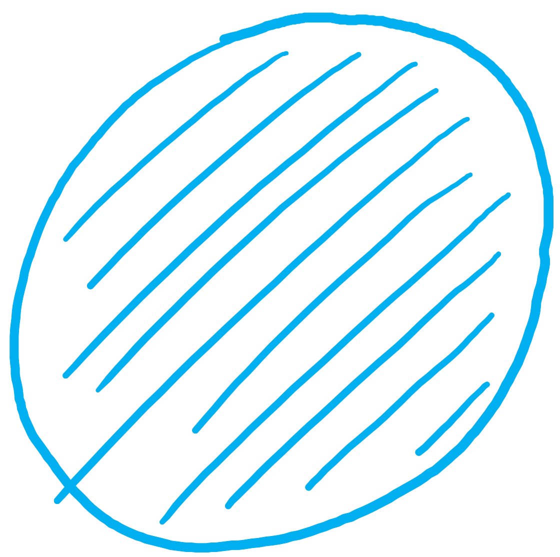
with singularities = critical points

$= \mathrm{Zero}(\phi) \cup \mathrm{Pole}(\phi)$.

☆ For CY-3 (BS case)

- simple zeros $z^1 dz^{\otimes 2}$
- (higher order) poles $z^{-k} dz^{\otimes 2}$

near smooth point

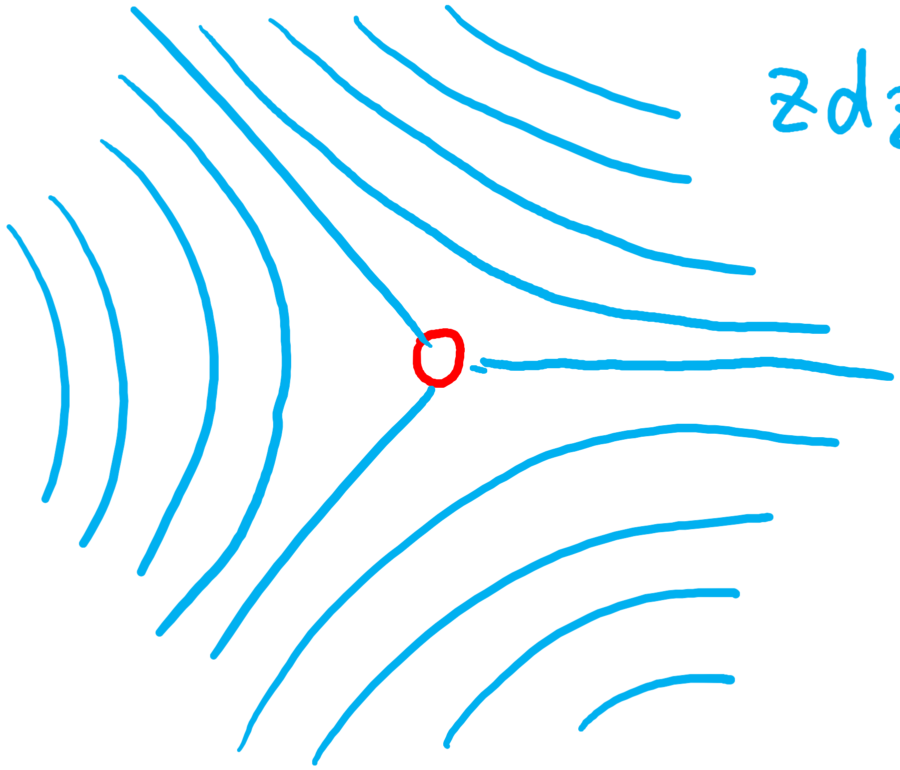


$\mathrm{Im}\, z$

"

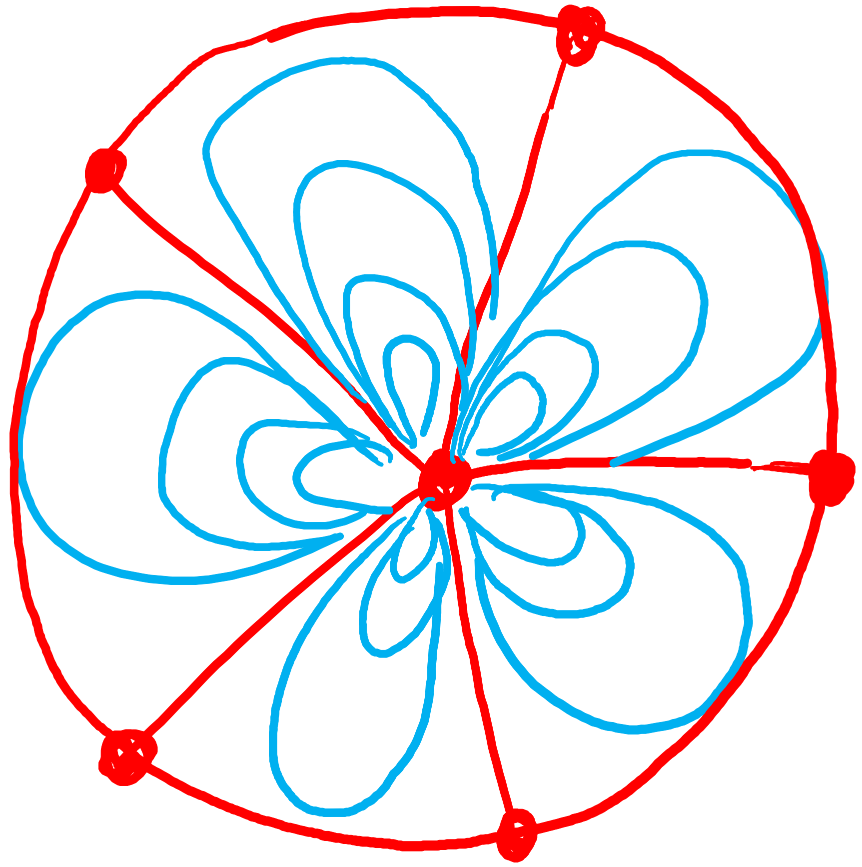
const.

in $\mathbb{C}$.



$z\,dz^{\otimes 2}$

$z \in \mathrm{Zero}$



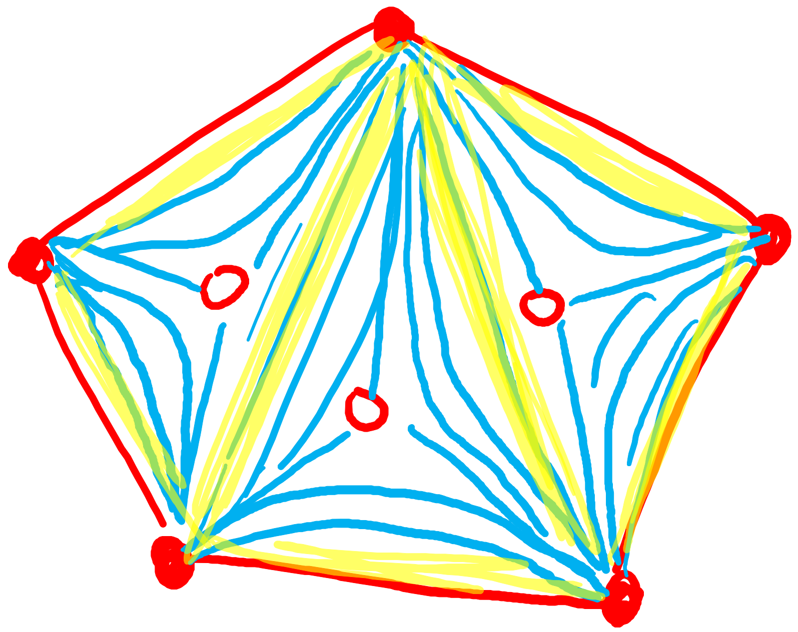
real blow-up

$z^{-7}dz^{\otimes 2}$

$p \in \mathrm{Pole} \rightsquigarrow \partial_p$

e.g. On $\mathbb{CP}^1$: ϕ of singularities type $(1,1,1,-7)$

$\rightsquigarrow$ after blow-up



<u>Key</u>: $\boxed{z\,dz^{\otimes 2}}$ $\boxed{z^{-k} dz^{\otimes 2}}$

$$\mathrm{FQuad}_3(\mathbb{S}) \cong \mathrm{Stab}^\circ \mathcal{D}_3(\mathbb{S}) \quad \text{BS}.$$

Ques: what's the relation?

☆ For CY-∞ (HKK case)

<u>Key</u> $\boxed{e^z dz^{\otimes 2}}$ $\boxed{e^{z^{-k}} \cdot z^{-\ell} dz^{\otimes 2}}$

$$\mathrm{FQuad}_\infty(\mathbb{S}) \cong \mathrm{Stab}^\circ \mathcal{D}_\infty(\mathbb{S}) \quad \text{HKK}$$

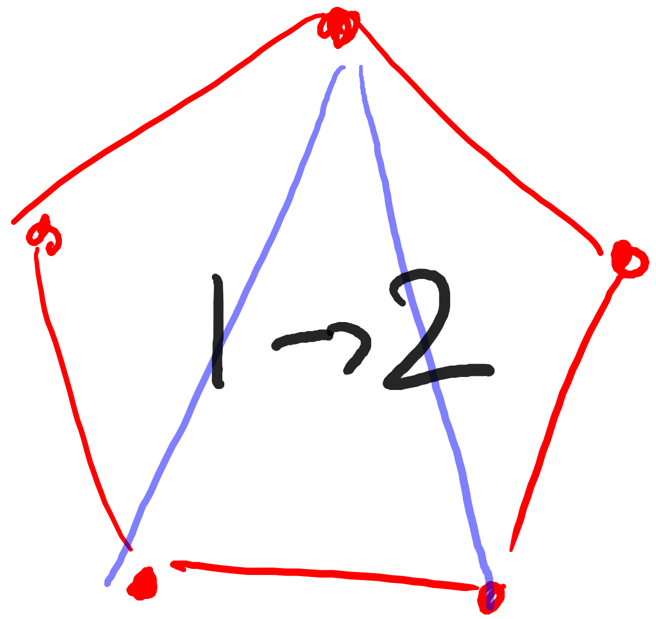
e.g. $\mathcal{D}_3(\mathbb{S}) = \mathcal{D}_3(Q)$, $\mathcal{D}_\infty(\mathbb{S}) = \mathcal{D}_\infty(Q)$

Construction of the categories:

$$\mathcal{D}_3(\mathbf{S}) = \mathcal{D}_{fd}(\Gamma(Q_T, W_T))$$

for T: a triangulation

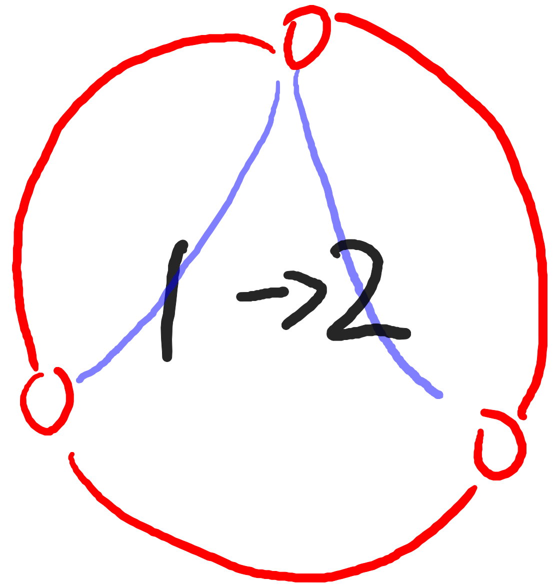
(Q, W): the ass. QP.



$$\mathcal{D}_\infty(\mathbf{S}) := \mathrm{TFuk}(\mathbf{S})$$

obj = arcs

morphism = intersections



Answer: ϱ-deformation.

Categorically $D_\infty \rightsquigarrow D_X \longrightarrow D_3$

CY-X completion $\qquad /\!/[x-3]$

$$e^z dz^{\otimes 2} \rightsquigarrow \boxed{z^{s-2} dz^{\otimes 2}} \rightsquigarrow s=3$$

$$e^{z^{-k}} \cdot z^{-\ell} dz^{\otimes 2} \longrightarrow \boxed{z^{-k(s-2)-\ell} dz^{\otimes 2}} \rightsquigarrow s=3$$

$s \in \mathbb{C}$ in ϱ-stab. cond.

Rem: $\ell = 2$ − winding number (k = order)

Thm (IQ) $\boxed{\mathrm{QStab}^{\circ}_{s}\mathcal{D}_X(\mathbb{S}) \cong \mathrm{QQuad}_s(\mathbb{S})}$

$\Downarrow$

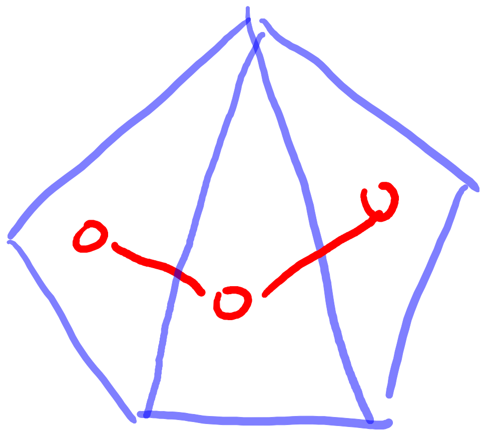
$s=3$ BS' result

& here $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 1$.

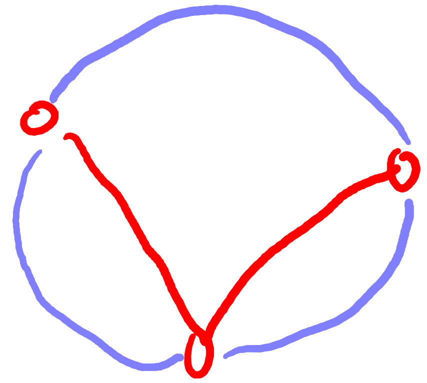
Thank you

Ideas

1° Decorate the marked surface in BS/FST setting



$\frac{BS}{FST}$ $\longleftrightarrow$



HKK

dual triangulation

= {saddle trajectories} = Core

2° Construct $\mathrm{QStab}_s \mathcal{D}_X$ from $\mathrm{Stab}\, \mathcal{D}_\infty$

On Grothendieck group: $K\mathcal{D}_X \cong K\mathcal{D}_\infty \otimes R$

Given $\sigma_\infty = (Z_\infty, P_\infty) \in \mathrm{Stab}\, \mathcal{D}_\infty$

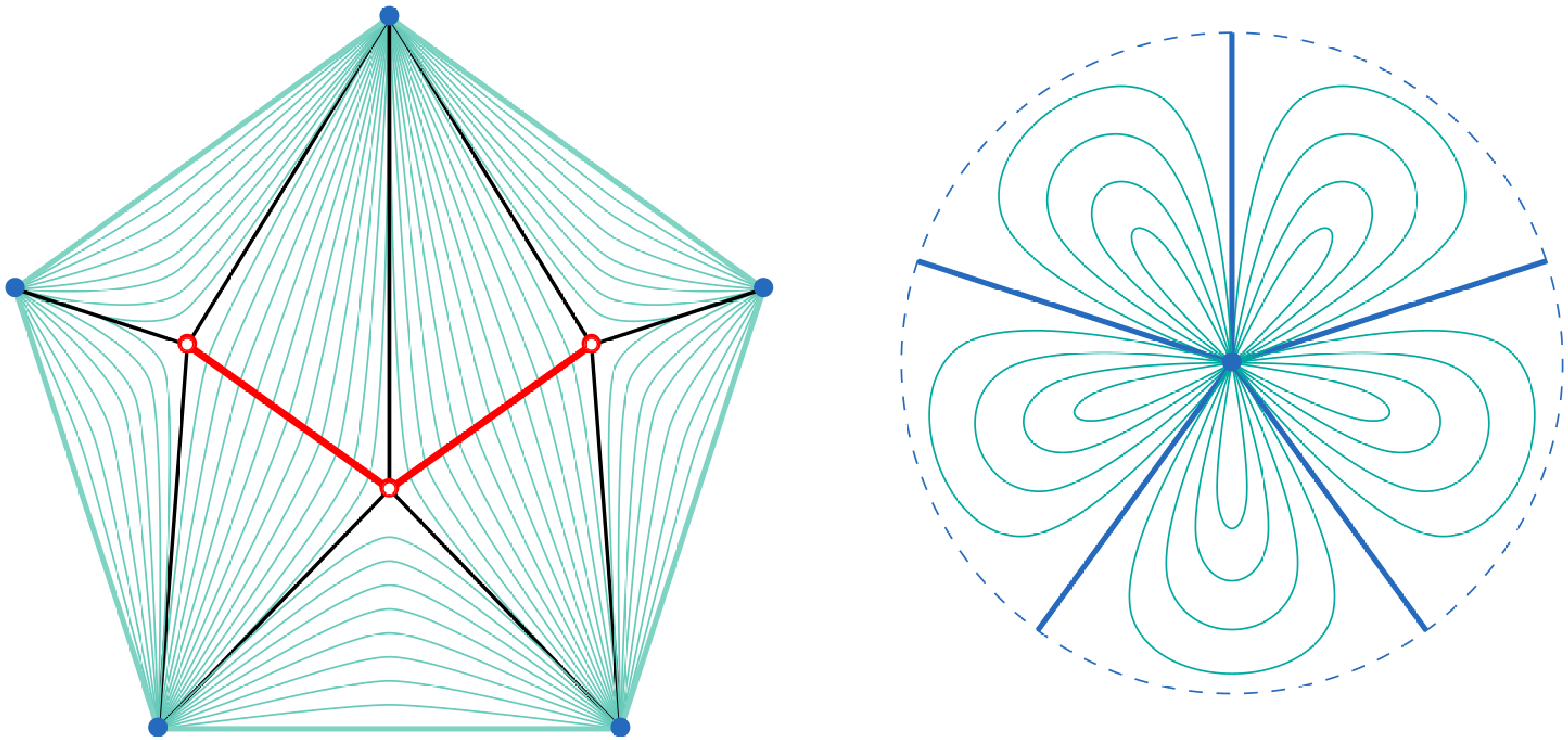
$$\begin{tikzcd}
Z_\infty : K\mathcal{D}_\infty \arrow[r] \arrow[d, "\otimes 1"'] & \mathbb{C} \arrow[d, "\otimes R"] \\
Z_\infty \otimes 1 : K\mathcal{D}_X \arrow[r] \arrow[dr, "Z"'] & \mathbb{C}[q^{\pm 1}] \arrow[d, "q \mapsto e^{i\pi s}"] \\
& \mathbb{C}
\end{tikzcd}$$

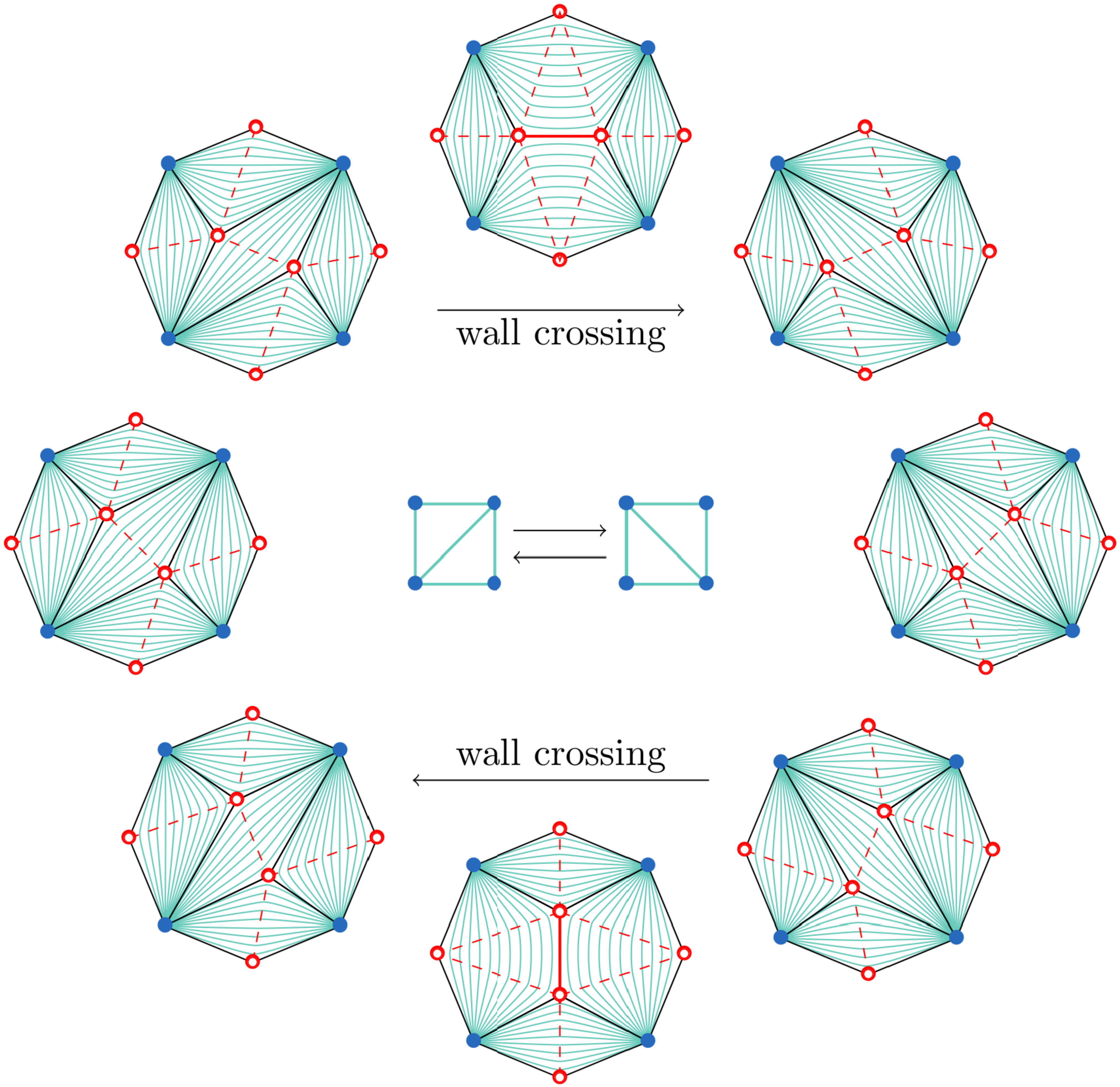
$$P(\phi) = \langle P_\infty(\phi - k\,\mathrm{Re}(s))[kX] \rangle$$

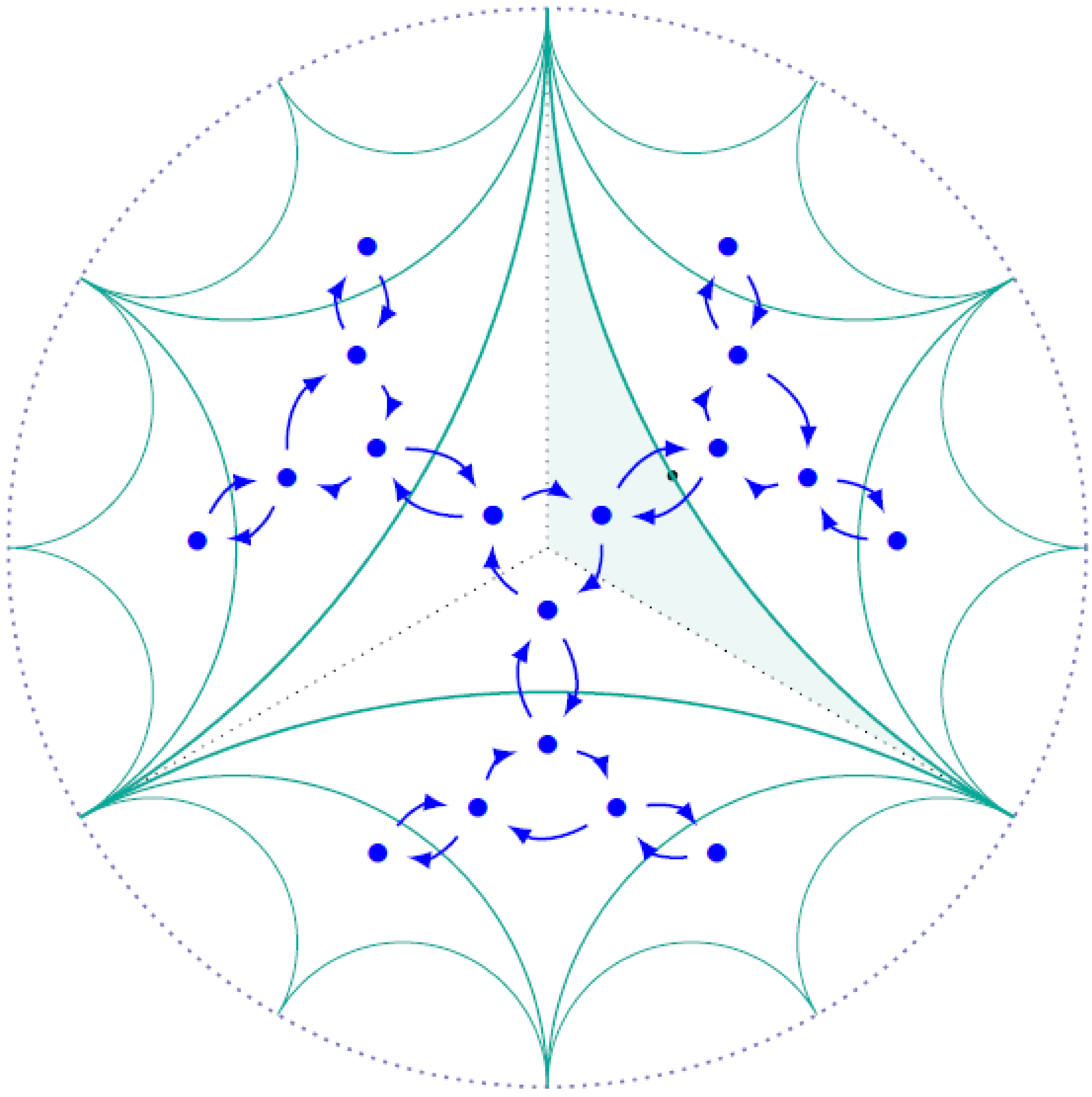
So we obtain $(Z, P) =: \sigma$

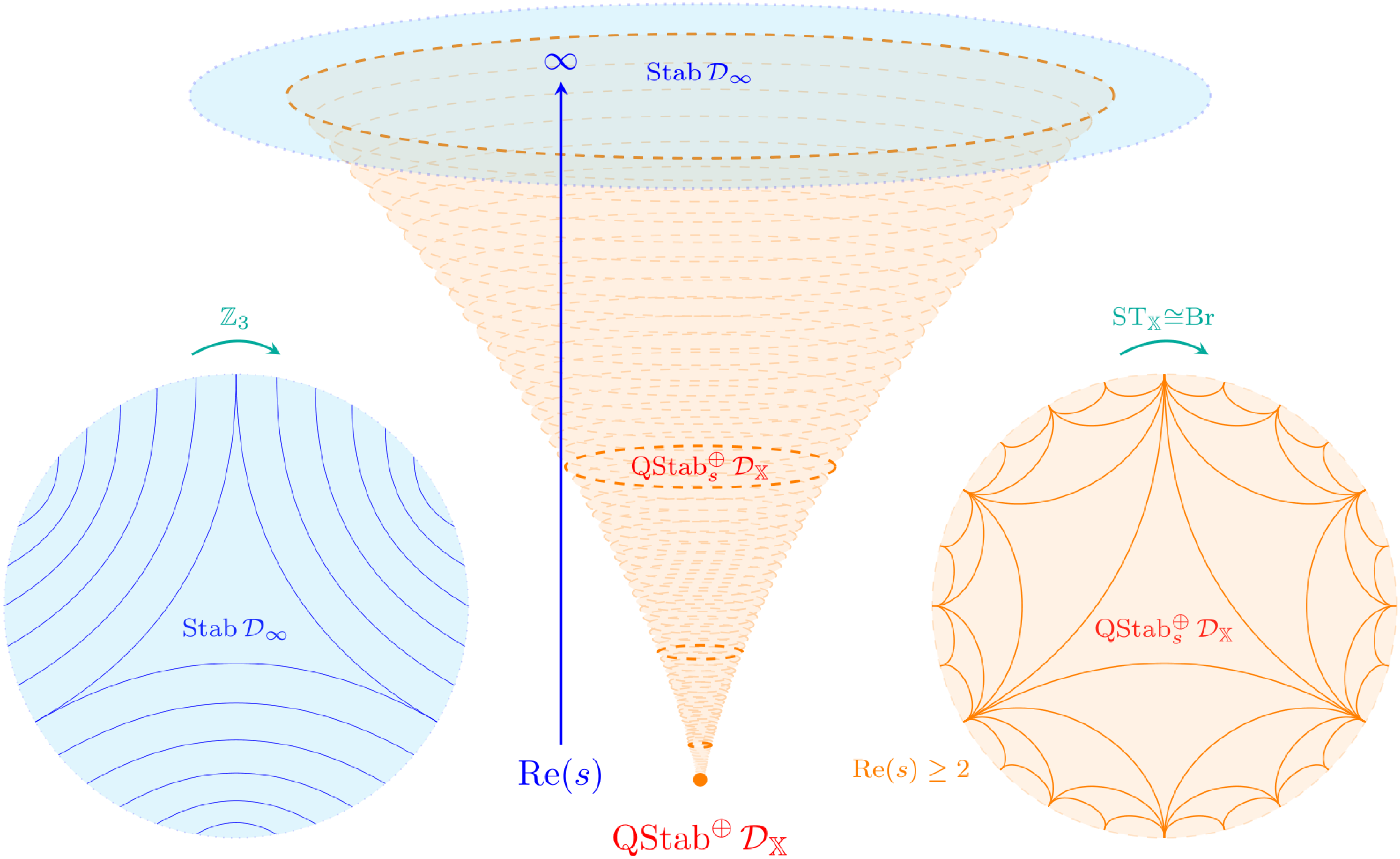
<u>Thm</u> (IQ) If $\boxed{\mathrm{Re}(s) \geq \mathrm{gldim}\, \sigma_\infty + 1}$, then $\sigma \in \mathrm{QStab}_s \mathcal{D}_X$.

$$\mathrm{gl.dim}\, P_\infty := \sup\{\phi_2 - \phi_1 \mid \mathrm{Hom}_{\mathcal{D}_\infty}(P(\phi_1), P(\phi_2)) \neq 0\}.$$




wall crossing
wall crossing





∞
$\mathrm{Stab}\,\mathcal{D}_\infty$
$\mathbb{Z}_3$
$\mathrm{ST}_{\mathbb{X}} \cong \mathrm{Br}$
$\mathrm{QStab}_s^{\oplus}\,\mathcal{D}_{\mathbb{X}}$
$\mathrm{Stab}\,\mathcal{D}_\infty$
$\mathrm{QStab}_s^{\oplus}\,\mathcal{D}_{\mathbb{X}}$
$\mathrm{Re}(s)$
$\mathrm{Re}(s) \geq 2$
$\mathrm{QStab}^{\oplus}\,\mathcal{D}_{\mathbb{X}}$