Wodzicki Characteristic Classes and Diffeomorphism Groups

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(with S. Egi and Y. Maeda)

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Outline

1. Motivation and Overview
2. What do we know about diffeomorphism groups?
3. Relating $\pi_1(\mathrm{Diff}(N))$ and $LN = \mathrm{Maps}(S^1, N)$
4. The geometry and topology of LN
5. Characteristic classes on TLN

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Motivation: Circle Actions

An action of S^1 on a closed manifold N, $a : S^1 \times N \to N$, $a(e^{i\theta}, n) = e^{i\theta} \cdot n$, gives a loop of diffeomorphisms $\ell(\theta) : N \to N, n \mapsto e^{i\theta} \cdot n$.

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Here the action is by isometries of S^{2k+1}, and the loop is the nonzero element of $\pi_1(SO(2k+1)) = \mathbb{Z}_2$.

Geometric quantization: Start with a symplectic manifold (M, ω) with $[\omega] \in H^2(M, \mathbb{Z})$. (So $\int_{\Sigma^2} \omega \in \mathbb{Z}$.) There exists a $\mathbb{C}$-line bundle $L \to M$ with connection, and with curvature $\Omega = \frac{1}{2\pi}\omega$. The *prequantum Hilbert space* is the space of sections $\Gamma(L)$.

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Partial Answer: Probably not at all, if this loop of diffeomorphisms of L^p is trivial in $\pi_1(\mathrm{Diff}(L^p))$.

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Theorem

Let (M^{4k}, ω) *be a closed integral symplectic manifold with associated line bundle* L*. For* $|p| \gg 0$*, rotating the fibers is an element of infinite order in* $\pi_1(\mathrm{Diff}(L^p))$ *and in* $\pi_1(\mathrm{Isom}(L^p))$.

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The proof will involve characteristic classes built from the Wodzicki residue on the tangent bundle of the loop space $L(L^p)$.

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- $\mathrm{Diff}(S^n) \not\sim O(n+1)$ for $n \geq 5$ [Smale];
  $\pi_i(\mathrm{Diff}(S^n)) \sim \pi_i(O(n)) \ltimes \Gamma$ with $|\Gamma| < \infty$ for $n \gg i$ [Farrell-Hsiang, 1970s]

  $\pi_1(\mathrm{Diff}(S^5)) = ?$

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Note: If $\pi_1(S^1 - \text{fiber}) \hookrightarrow \pi_1(L^p)$, as in this case, then an argument due to Mitsumatsu gives the result.

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Relating $\pi_1(\mathrm{Diff}(N))$ and LN

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$$\mathrm{Maps}(S^1 \times N, N) = \mathrm{Maps}(S^1, \underbrace{\mathrm{Maps}(N, N)}_{\mathrm{Diff}(N)\ \text{here}}) = \mathrm{Maps}(N, \underbrace{\mathrm{Maps}(S^1, N)}_{LN}).$$

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a^L induces $a^L_* : H_{4k+1}(N) \to H_{4k+1}(LM)$. Set $[a^L] = a^L_*[N] \in H_{4k+1}(LN)$.

Lemma

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The operators will be $\Psi\mathrm{DOs}$, and the trace will be the Wodzicki residue.

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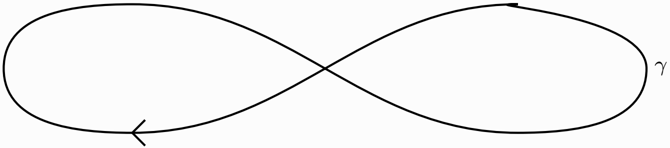

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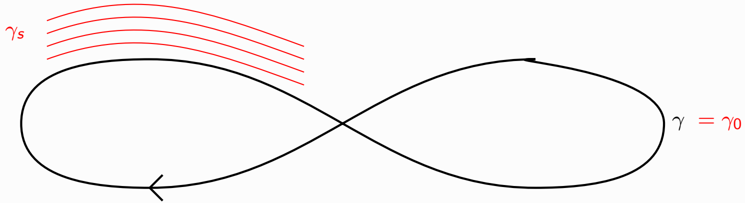

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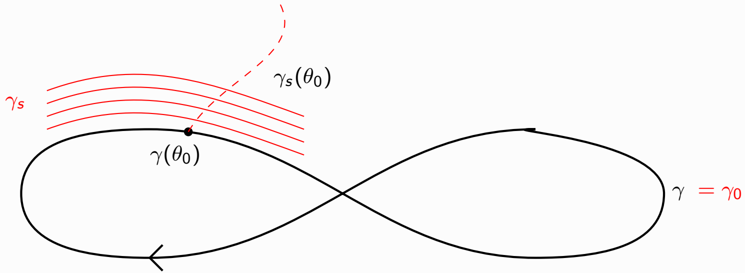

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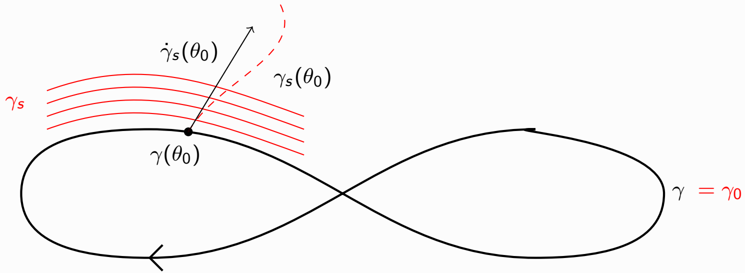

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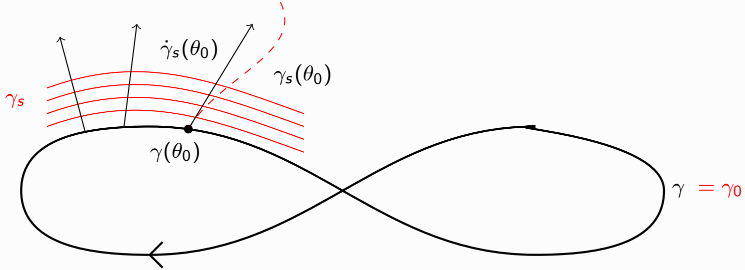

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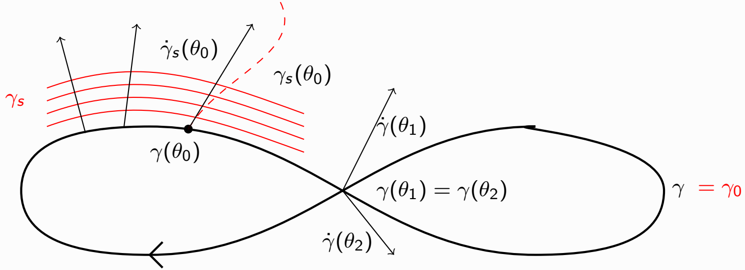

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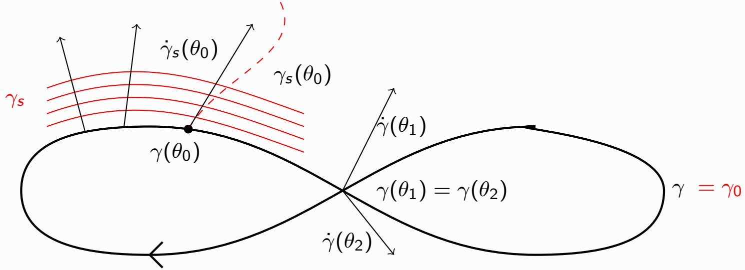

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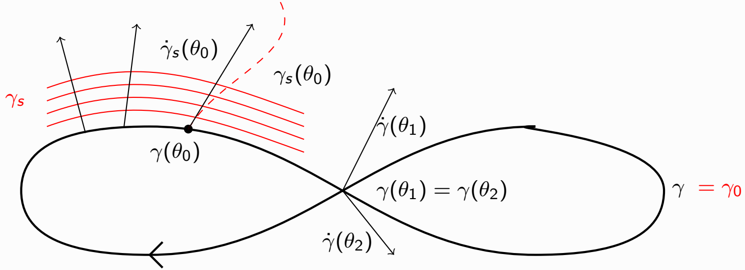

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The structure group of TN is $GL(n, \mathbb{R})$; the structure group of TLN is $\mathcal{G} = Maps(S^1, GL(n, \mathbb{R}))$, the gauge group of $\gamma^* TM = S^1 \times \mathbb{R}^n \to S^1$.

Natural metrics on LN

Fix a loop $\gamma \in LN$. A tangent vector $X \in T_\gamma LN = \Gamma(\gamma^* TN \to S^1)$ is

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Put a Riemannian metric $\langle\ ,\ \rangle$ on N. The L^2 inner product on LN is

$$\langle X, Y\rangle_0 = \frac{1}{2\pi}\int_{S^1} \langle X(\theta), Y(\theta)\rangle_{\gamma(\theta)} d\theta, \quad X, Y \in \Gamma(\gamma^* TN).$$

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Fix a loop $\gamma \in LN$. A tangent vector $X \in T_\gamma LN = \Gamma(\gamma^* TN \to S^1)$ is

$$X : \theta \in S^1 \mapsto X(\theta) \in T_{\gamma(\theta)}N.$$

Put a Riemannian metric $\langle \; , \; \rangle$ on N. The L^2 inner product on LN is

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A stronger metric is given by picking a Sobolev parameter $s \geq 0$. The s-inner product on $T_\gamma LN$ is

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Here $\Delta = D^* D$, $D = \frac{D}{d\dot{\gamma}}$, the covariant derivative along γ.

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Think of s as an annoying regularization parameter. Meaningful results should be independent of s.

Natural connections on LN

The Sobolev-s metric makes LN a Riemannian manifold. The Levi-Civita connection ∇^s on LN is determined by

$$\begin{aligned}\langle \nabla^s_Y X, Z\rangle_s &= X\langle Y, Z\rangle_s + Y\langle X, Z\rangle_s - Z\langle X, Y\rangle_s \\ &\quad +\langle [X, Y], Z\rangle_s + \langle [Z, X], Y\rangle_s - \langle [Y, Z], X\rangle_s.\end{aligned}$$

since the right hand side is a *continuous* linear functional of $Z \in T_\gamma LN = \Gamma(\gamma^* TN)$ (for the right topology on the space of sections).

For $s = 0$, the Levi-Civita connection ∇^0 is just the integrated version of the Levi-Civita connection ∇^N on N for its metric $\bar{g}$.

$$\begin{aligned}\langle \nabla^0_X Y, Z\rangle_{\gamma, L^2} &= \int_\gamma \langle \nabla^N_X Y(\theta), Z(\theta)\rangle_{\gamma(\theta),\bar{g}} d\theta \\ &= \int_\gamma \langle X(Y) + \omega^N_X(Y), Z\rangle_{\gamma(\theta),\bar{g}} d\theta.\end{aligned}$$

So

$$\begin{aligned}\nabla^0_X Y &= X(Y) + \text{connection one} - \text{form} \\ &= X(Y) + \text{a bundle endormophism of } T_\gamma LN.\end{aligned}$$

Natural connections on LN

Let $\bar{R}$ be the curvature tensor of $\bar{g}$.

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Proposition

The $s = 1$ Levi-Civita connection is given by

$$\begin{aligned}
&\nabla^1_X Y(\gamma)(\theta) \\
&= \nabla^0_X Y(\gamma)(\theta) + \frac{1}{2}(1+\Delta)^{-1}\left[-\nabla_{\dot\gamma}(\bar{R}(X,\dot\gamma)Y)(\theta)\right. \\
&\qquad -\bar{R}(X,\dot\gamma)\nabla_{\dot\gamma}Y(\theta) - \nabla_{\dot\gamma}(\bar{R}(Y,\dot\gamma)X)(\theta) - \bar{R}(Y,\dot\gamma)\nabla_{\dot\gamma}X(\theta) \\
&\qquad \left.+(\bar{R}(X,\nabla_{\dot\gamma}Y)\dot\gamma)(\theta) + (\bar{R}(Y,\nabla_{\dot\gamma}X)\dot\gamma)(\theta)\right].
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$$\nabla^1_X Y = \underbrace{[X(Y) + \text{a bundle endomorphism}]}_{\Psi\text{DO of order } 0} + [\text{a } \Psi\text{DO of order } -1].$$

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The connection 1-form and curvature 2-form $\omega^1_X \in End(T_\gamma LN) = End(\Gamma(\gamma^* TN)), \Omega^1 = d\omega^1 + \omega^1 \wedge \omega^1$ are *zeroth order* Ψ*DOs* acting on $Y \in T_\gamma LN = \Gamma(\gamma^* TN \to S^1)$.

ΨDOs

Let $\Omega \subset \mathbb{R}^n$ be a precompact domain.

For
$$\partial^\alpha = (\partial_{x^1})^{\alpha_1} \cdot \ldots \cdot (\partial_{x^n})^{\alpha_n}, \ \xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n},$$
let $D = \sum_{|\alpha| \leq n_0} a_\alpha(x) \partial^\alpha : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ be a differential operator.

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By Fourier transform and Fourier inversion,

$$Df(x) = \sum_{|\alpha| \leq n_0} a_\alpha(x)(\partial^\alpha)(x) == \int_{T^*\Omega} e^{i(x-y)\cdot\xi}\sigma_D(x,\xi)f(y)\,dy\,d\xi$$

where the *symbol* of D is the polynomial $\sigma_D(x,\xi) = \sum_{|\alpha| \leq n_0} \frac{1}{i^{|\alpha|}} a_\alpha(x)\xi^\alpha$. $\sigma_D \sim |\xi|^{n_0}$ as $|\xi| \to \infty$. ΨDOs are defined by the same integral, but with symbol

$$\sigma(x,\xi) \sim \sum_{k \in \mathbb{Z}_{\geq 0}} a_{n_0-k}(x)\xi^{n_0-k}$$

growing like $|\xi|^{n_0}$, where the order n_0 of D can be any real number.

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Slight generalization to vector valued linear operators:
$D = \sum_{|\alpha| \leq n_0} a_\alpha(x) \partial^\alpha : C_c^\infty(\Omega, \mathbb{R}^m) \to C_c^\infty(\Omega, \mathbb{R}^n)$ with $a_\alpha(x) \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$.

This extends to linear operators on manifolds and then to operators on sections of bundles $E \to N$ over closed manifolds. For $x \in N, \xi \in T^*N$, $\sigma(x, \xi) \in \mathrm{Hom}(E_x, E_x)$.

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D is *elliptic* if $\sigma_{n_0}(x,\xi)$ is invertible for $\xi \neq 0$. Standard Laplacian operators are elliptic, with top symbol $\sigma_2(\Delta)(x,\xi) = |\xi|^2\mathrm{Id}$, as are their inverses (Green's operators) like $(1+\Delta)^{-1}$, with top symbol $\sigma_{-2}(\Delta^{-1})(x,\xi) = |\xi|^{-2}\mathrm{Id}$.

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Just like DO, $\Psi\mathrm{DO}(E)$ forms a graded algebra, and includes all Green's operators, heat operators, and operators given by smooth kernels. Powers of elliptic operators, like $(1+\Delta)^s$, are again $\Psi\mathrm{DOs}$.

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$$\mathrm{res}^W(AB) = \mathrm{res}^W(BA).$$

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$\mathrm{res}^W : \Psi\mathrm{DO}_{\leq 0} \to \mathbb{C}$ is Ad-invariant, so for a $\Psi\mathrm{DO}_0^*$-connection on TLM with curvature Ω, we have

Definition:

(i) The i$^{\mathrm{th}}$ *Wodzicki-Chern character class* of LN is

$$\begin{aligned} ch_i^W(LN) &= [\mathrm{res}^W(\Omega^i)] \\ &= \left[\int_{S^*S^1} \mathrm{tr}_x(\sigma_{-1}(\Omega^i)(x,\xi))\ d\xi\ dx\right] \\ &\quad \in H^{2i}(LN, \mathbb{C}). \end{aligned}$$

Chern-Simons classes on TLN

Problem: $ch_i^W(LN) = 0$. Since $ch_i^W(LN)$ is independent of connection, we can compute it for the L^2 connection ∇^0:

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If Chern character forms vanish for two connections ∇_0, ∇_1 on $E \to N$, then Chern-Simons classes are defined: there is an explicit/pointwise computable form $CS_i \in \Lambda^{2i-1}(N)$ with

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If e.g. ∇_0, ∇_1 are flat, or if $\dim(N) = 2i - 1$, then $CS_i(\nabla_0, \nabla_1)$ is closed and defines the Chern-Simons class

$$CS_i(\nabla_0, \nabla_1) \in H^{2i-1}(N, \mathbb{C}).$$

WCS classes on TLN

LN is infinite dimensional, but the local nature of res^W implies $ch_i^W(\Omega^s) \equiv 0$ as a form if $\dim N = 2i - 1$.

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CS_i^W involves res^W, so it is locally computable.

Proposition

At a loop $\gamma \in LN$,

$$\begin{aligned} &CS_{2i-1}^W(X_1, \ldots, X_{2i-1})(\gamma) \\ &= \frac{i}{2^{i-2}} \sum_\sigma \mathrm{sgn}(\sigma) \int_\gamma \mathrm{tr}[(\bar{R}(X_{\sigma(1)}, \cdot)\dot{\gamma})(\bar{\Omega}^N)^{i-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2i-1)})], \end{aligned}$$

where $\bar{R}, \bar{\Omega}$ are the curvature tensor and curvature two-form on N.

WCS forms on LN for $N = L^p$

We have $S^1 \to L^p \to (M, \omega)$. Take an almost complex structure J and a Riemannian metric g on M compatible with ω. Using the connection on L^p associated to $p\omega$, we get a metric g_p on L^p.

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Lemma

Let X, Y, Z, W be tangent vectors to (M, ω, g, J), let R be the curvature of M, let X^L, etc. be their horizontal lifts to $N = L^p$, let $\bar{R}$ be the curvature of N, and let ξ be the tangent vector along the circle fiber. Then

$$\begin{aligned} g_p(\bar{R}(X^L, Y^L)Z^L, W^L) &= g(R(X,Y)Z, W) + p^2[-g(JY,Z)g(JX,W) \\ &\quad + g(JX,Z)g(JY,W) + 2g(JX,Y)g(JZ,W)], \\ g_p(\bar{R}(X^L, Y^L)Z^L, \bar{\xi}) &= -pg((\nabla_X J)Y, Z) + pg\left((\nabla_Y J)X, Z\right), \\ &\text{etc.} \end{aligned}$$

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We have $S^1 \to L^p \to (M, \omega)$. Take an almost complex structure J and a Riemannian metric g on M compatible with ω. Using the connection on L^p associated to $p\omega$, we get a metric g_p on L^p.

Lemma

Let X, Y, Z, W be tangent vectors to (M, ω, g, J), let R be the curvature of M, let X^L, etc. be their horizontal lifts to $N = L^p$, let $\bar{R}$ be the curvature of N, and let ξ be the tangent vector along the circle fiber. Then

$$\begin{aligned} g_p(\bar{R}(X^L, Y^L)Z^L, W^L) &= g(R(X,Y)Z,W) + p^2[-g(JY,Z)g(JX,W) \\ &\quad + g(JX,Z)g(JY,W) + 2g(JX,Y)g(JZ,W)], \\ g_p(\bar{R}(X^L, Y^L)Z^L, \bar{\xi}) &= -pg((\nabla_X J)Y, Z) + pg\left((\nabla_Y J)X, Z\right), \\ &\text{etc.} \end{aligned}$$

Plugging this into

$$CS^W_{2i-1}(X_1, ..., X_{2i-1})(\gamma) = \frac{i}{2^{i-2}} \sum_\sigma \mathrm{sgn}(\sigma) \int_\gamma \mathrm{tr}[(\bar{R}(X_{\sigma(1)}, \cdot)\dot{\gamma})(\bar{\Omega}^N)^{i-1} ...,$$

we get:

End of the proof

As a $2i-1$ form on $L(L^p)$,

$$CS^W_{2i-1} = \sum_{j=1}^{i} \alpha_j p^{2j}.$$

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Let $\dim(N) = 4k+1$. *Then* $[a^D] \in \pi_1(\mathrm{Diff}(N))$ *has infinite order iff* $0 \neq [a^L] \in H_{4k+1}(LN)$ *iff* $\int_{[a^L]} \alpha \neq 0$, *where* α *is a closed* $(4k+1)$*-form on* LN.

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For $\alpha = CS^W_{4k+1}$ on $L^p \to M^{4k}$,

$$\int_{[a^L]} \alpha = \int_{L^p} a^{L,*} CS^W_{4k+1} = \sum_j \left(\int_{L^p} a^{L,*} \alpha_j \right) p^{2j} \neq 0 \text{ for } p \gg 0$$

iff

$$\int_{L^p} a^{L,*} \alpha_j \neq 0 \text{ for some } j.$$

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Lemma

For $\dim(M) = 4k$*, and* $CS^W_{4k+1} = \sum_{j=1}^{2k+1} \alpha_j p^{2j}$*, the highest term*

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Thus $\pi_1(\mathrm{Diff}(N))$ has infinite order for $N = L^p$ with $p \gg 0$. □

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Thus $\pi_1(\mathrm{Diff}(N))$ has infinite order for $N = L^p$ with $p \gg 0$. $\square$

Remark: $a^{L,*}\alpha_1 = (cons.(\text{Pontrjagin form of degree } 4k) \wedge d\xi) + \left(\beta \text{ with } \int_{L^p} \beta = 0\right)$.
What is the geometric significance of the other $a^{L,*}\alpha_j$?

Explicit Calculations

Kähler Example: There is a family of Sasaki-Einstein metrics $g_a, a \in (0,1)$, on B^5 which match up nicely on ∂B^5 to give metrics on $N = S^2 \times S^3 \to M = S^2 \times S^2$. We get

$$\int_N a^{L,*} CS_5^W(g_a) = -\frac{1849\pi^4}{37750}(-1 + a^2),$$

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Non-Kähler Example: On the Kodaira-Thurston example $T^2 \to M^4 \to T^2$, there is an explicit metric such that

$$\int_{L^p} CS_5^W = 10(-1 - 24p^2 + 288p^4) - 3\sqrt{5}(1 + 64p^2)\coth^{-1}\sqrt{5}.$$

Thus $\pi_1(\mathrm{Diff}(L^p))$ is infinite for all p.

Thank you!