Elliptic Chiral Homology and Quantum Master Equation

Si Li

YMSC, Tsinghua University

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Motivation

Given a deformation quantization $\mathcal{A}_\hbar(M) = (C^\infty(M)[[\hbar]], \star)$ on a symplectic manifold (X, ω), there exists a unique linear map

$$\mathrm{Tr} : C^\infty(M)[[\hbar]] \to \mathbb{C}((\hbar))$$

satisfying a normalization condition and the trace property

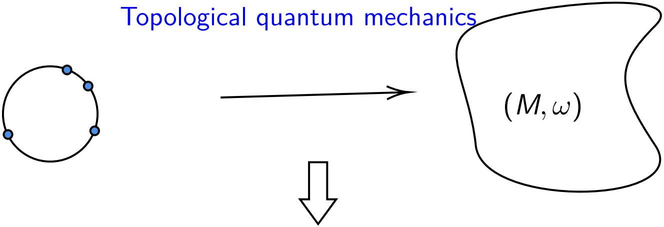
$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f).$$

Then

$$\mathrm{Tr}(1) = \int_M e^{\omega/\hbar} \hat{A}(M).$$

This is the algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of Atiyah-Singer index theorem.

In [**Grady-Li-L 2017**] [**L-Xu-Gui, 2020**] A rigorous connection between the effective BV quantization for topological quantum mechanics and the algebraic index theorem.

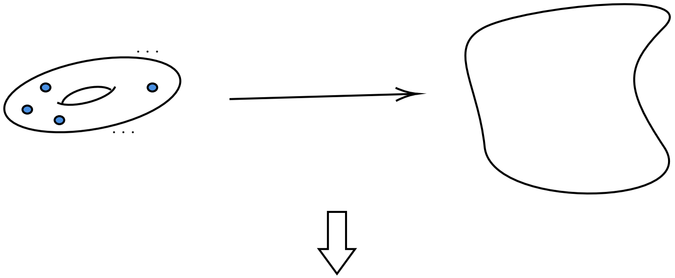


Algebraic index theory

$$\mathrm{Tr} : HH_\bullet(\mathcal{A}_\hbar(M)) \to \mathbb{C}(\!(\hbar)\!)$$

$$\int_{BV} \int_{\mathrm{Conf}_m(S^1)} dt_1 \cdots dt_m \left\langle \mathcal{O}_0(t_0)\mathcal{O}_1^{(1)}(t_1) \ \cdots \mathcal{O}_m^{(1)}(t_m) \right\rangle$$

Replace S^1 by an elliptic curve E. (**Witten**: index of dirac operators on loop space).



2d Chiral analogue of algebraic index?

Need to study chiral deformation of 2d conformal field theories.

Mirror symmetry

Mirror symmetry is about a duality between

symplectic geometry (A-model) $\Longleftrightarrow$ complex geometry (B-model)

$$
\begin{array}{ccc}
\int_{\mathrm{Map}(\Sigma_g, X)} \text{(A-model)} & \overset{\text{Fourier transform}}{=\!=\!=\!=} & \int_{\mathrm{Map}(\Sigma_g, X')} \text{(B-model)} \\
\Big\downarrow \text{localize} & & \text{localize} \Big\downarrow \\
\int_{\text{Holomorphic maps}(\Sigma_g, X)} & \leftarrow - - - - \rightarrow & \int_{\text{Constant maps}(\Sigma_g, X')} \ ??? \\
\Big\Downarrow & & \Big\downarrow \\
\text{Gromov-Witten Theory} & & \text{Hodge theory}
\end{array}
$$

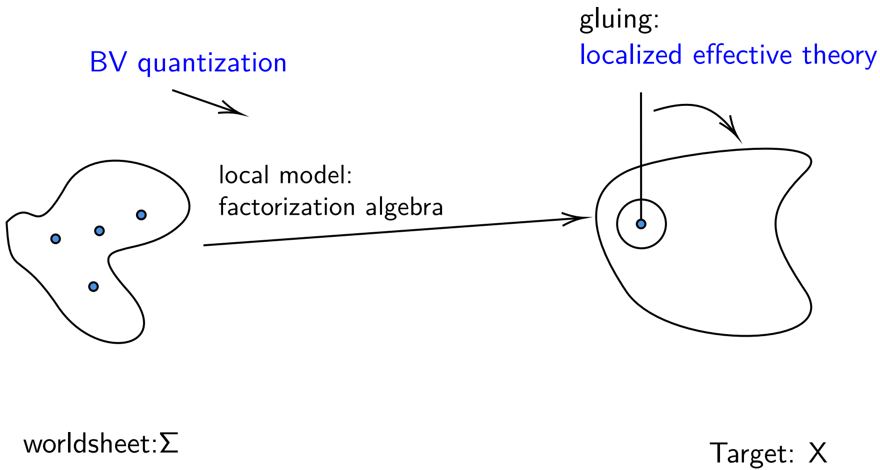
The B-model can be viewed as a suitable mysterious way to

"count constant surfaces"

related to the variation of Hodge structures and its quantization.

We will be mainly interested in σ-models about the mapping space

$$\varphi : \Sigma \to X$$



Family of QFT glued via **Gelfand-Kazhdan** formal geometry.

Two models

$$\varphi : \Sigma \to X$$

1. Topological quantum mechanics

$$\dim \Sigma = 1$$

2. Chiral deformation of CFT.

$$\dim \Sigma = 2$$

Several examples in the literature fit into these lines:

- **Kontsevich** and **Cattaneo-Felder**: Poisson σ-model.
- **Malikov-Schechtman-Vaintrob**: Chiral de Rham complex
- **Costello**: holomorphic CS theory on an elliptic curve E
- **Grady-Gwilliam**: TQM on $X = T^*M$
- **Grady-Li-L**: TQM on a symplectic manifold X
- **Gorbounov-Gwilliam-Williams**: $\beta - \gamma$-system on $X = T^*M$.
- ...

BV algebra

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- $\Delta : \mathcal{A} \to \mathcal{A}$ is a linear operator of degree 1 such that $\Delta^2 = 0$.
- The **BV bracket** $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|} a \Delta b, \;\; a, b \in \mathcal{A}.$$

Then $\{-,-\}$ satisfies the following graded Leibniz rule

$$\{a, bc\} := \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}, \quad a, b, c \in \mathcal{A}.$$

Quantum master equation

Let $(C_\bullet, d)$ be a chain complex over $\mathbb{C}[[\hbar]]$. A $\mathbb{C}[[\hbar]]$-linear map

$$\langle - \rangle : C_\bullet \to \mathcal{A}((\hbar))$$

is said to satisfy quantum master equation (QME) if

$$\boxed{(d + \hbar\Delta)\langle - \rangle = 0.}$$

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Example

Let $(C_\bullet, d) = (\mathbb{C}[\![\hbar]\!], 0)$ and $I = I_0 + I_1\hbar + \cdots \in \mathcal{A}[\![\hbar]\!]$.

$$\langle c \rangle := c e^{I/\hbar}$$

satisfies QME if and only if $\hbar\Delta I + \frac{1}{2}\{I, I\} = 0$.

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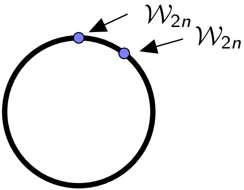
- Partition function: $\quad \mathrm{Index} = \int_{BV} \langle 1 \rangle.$

Example: 1d TQM



- Local observables: Weyl algebra

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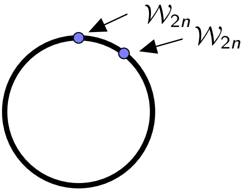

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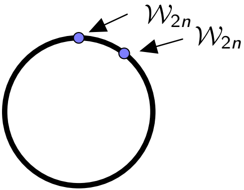

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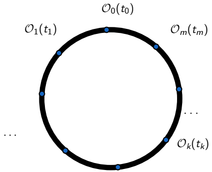
$$\mathrm{Obs}_{1d} = \mathcal{W}_{2n} = (\mathbb{C}[\![p_i, q^i]\!][\![\hbar]\!], \star)$$

- $(C_\bullet(\mathrm{Obs}_{1d}), b)$ = the Hochschild chain complex.

- BV algebra $(\mathcal{A}_{1d}, \Delta) = (\mathbb{C}[\![y^i, dy^i]\!], \mathcal{L}_\Pi)$. Here Π = Poisson.



$$\mathrm{Conf}_m(S^1) = \{t_1, \cdots, t_n \in S^1 | t_i \neq t_j\}$$

- $\langle - \rangle_{1d} : C_\bullet(\mathcal{W}_{2N}) \to \mathcal{A}_{1d}((\hbar))$ where

$$\langle \mathcal{O}_0 \otimes \mathcal{O}_1 \cdots \otimes \mathcal{O}_m \rangle_{1d} \qquad \mathcal{O}_i \in \mathcal{W}_{2n}$$

$$= \int_{\mathrm{Conf}_m(S^1)} dt_1 \cdots dt_m \left\langle \mathcal{O}_0(t_0)\mathcal{O}_1^{(1)}(t_1) \ \cdots \mathcal{O}_m^{(1)}(t_m) \right\rangle_{free}$$

Here $\mathcal{O}_i^{(1)}(t)dt$ is the topological descend of $\mathcal{O}_i(t)$. It satisfies

$$\text{QME} \qquad (b + \hbar\Delta)\langle - \rangle_{1d} = 0$$

Here b is the Hochschild differential.

Ref: [**L-Xu-Gui**, 2020]

This construction can be glued on a symplectic target X

$$
\begin{array}{c}
W(X) := Fr(X) \times_{Sp_{2n}} \mathcal{W}_{2n} \\
\downarrow \\
X
\end{array}
$$

which carries a flat connection (Fedosov connection)

$$D = d + \frac{1}{\hbar}[\gamma, -]_\star, \quad D^2 = 0.$$

Here $\gamma \in \Omega^1(X, W(X))$. Fedosov connection is the geometric interpretation of quantum master equation [**Grady-Li-L** 2017].

$\langle - \rangle_{1d}$ leads to a trace map on deformation quantized algebra, as explicitly described by [**Feigin-Felder-Shoikhet**, 2003].

Chiral CFT and Chiral Index

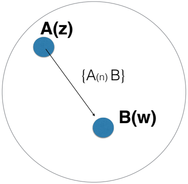
Vertex operator algebras

A *vertex algebra* is a vector space $\mathcal{V}$ with the structure of state-field correspondence (and other axioms like vacuum, locality, etc.)

$$\mathcal{V} \to End(\mathcal{V})[[z, z^{-1}]]$$
$$A \to A(z) = \sum_n A_{(n)} z^{-n-1}$$

We ofter write $Y(A, z)$ for $A(z)$ for the corresponding operator.
It defines the operator product expansion (OPE)

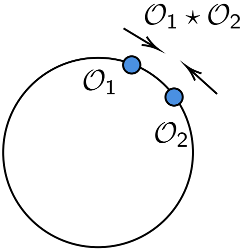
$$A(z)B(w) = \sum_{n \in \mathbb{Z}} \frac{(A_{(n)} \cdot B)(w)}{(z-w)^{n+1}}$$



Free CFT's give rise to examples of vertex algebras $\mathcal{V}$.

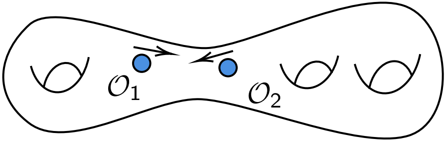
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$

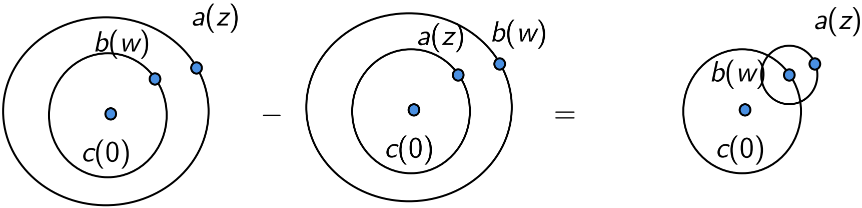


The Borcherds identity

We have

$$\sum_{j\geq 0}\binom{m}{j}(a_{(n_j)}b)_{(m+n-j)}c$$

$$=\sum_{j\geq 0}(-1)^j\binom{n}{j}(a_{(m+n-j)}b_{(k+j)}c-(-1)^n b_{(n+k-j)}a_{(m+j)}c.)$$



VOA examples: $\beta\gamma - bc$ system.

Let $\mathbf{h} = \mathbf{h}_{\bar{0}} \oplus \mathbf{h}_{\bar{1}}$ be equipped with an even symplectic pairing

$$\langle -, - \rangle : \wedge^2 \mathbf{h} \to \mathbb{C}$$

We obtain a vertex algebra structure on the free differential ring

$$\mathcal{V}^{\beta\gamma - bc}(\mathbf{h}) \cong \mathbb{C}[\![\partial^k a^i]\!], \quad a^i \text{ is a basis of } \mathbf{h}, k \geq 0$$

The OPE's are generated by

$$a(z)b(w) \sim \hbar \frac{\langle a, b \rangle}{(z-w)}, \quad \forall a, b \in \mathbf{h}.$$

A chiral σ-model

$$\varphi : \Sigma \to X$$

will produce a bundle $\mathcal{V}(X)$ of chiral vertex operator algebras

$$\begin{array}{c} \mathcal{V}(X) \\ \big\downarrow \\ X \end{array}$$

This is the chiral analogue of Weyl bundle in TQM.

Theorem (L, 2016)

The quantization of the 2d chiral model is equivalent to solving a flat connection on the vertex algebra bundle $\mathcal{V}(X)$

$$D = d + \frac{1}{\hbar}\left[\oint \mathcal{L}, -\right], \quad D^2 = 0$$

where $\mathcal{L} \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \mathcal{L}$ is the associated chiral vertex operator fiberwise.

- This is the chiral analogue of Fedosov connection.
- The quantization is formulated in the BV formalism.
- BRST reduction of chiral models falls into this setup

$$\oint \mathcal{L} = \text{BRST operator.}$$

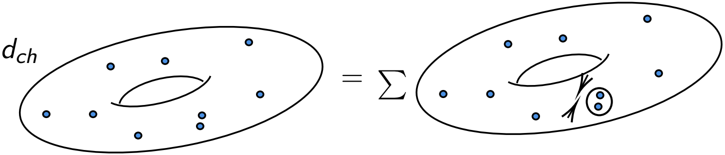
Elliptic chiral homology

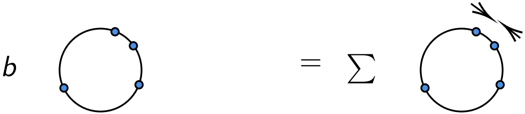
- In [**Zhu**, 1994], **Zhu** studied the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.
- **Beilinson** and **Drinfeld** define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.
- Recently, [**Ekeren-Heluani**,2018,2021]: an explicit complex expressing the 0th and 1st elliptic chiral homology.

We now review the construction of **Ekeren-Heluani**, which fits particularly well in our setup.

Elliptic chiral homology

Intuitively, the chiral differential in the chiral complex can be viewed as a 2d chiral analogue of the Hochschild differential b.





$$b(a_0 \otimes \cdots \otimes a_p)$$

$$= (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_p.$$

For each n we denote by $\mathcal{F}_n$ the ring of meromorphic functions $f(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ such that

$$f(z_1, \ldots, z_i + m + l\tau, \ldots, z_n) = f(z_1, \ldots, z_n), m, l \in \mathbb{Z}$$

and f is allowed to have poles at

$$z_i - z_j = m + l\tau, 1 \leq i \neq j \leq n, m, l \in \mathbb{Z}.$$

Equivalently, let $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. Then $\mathcal{F}_n$ are meromorphic functions on E_τ^n with possible poles along diagonals.

An explicit chain complex by **Ekeren-Heluani**

Genus 1 chiral homology of a VOA $\mathcal{V}$ can be computed via

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

where

$$C_0(\mathcal{V}) = \mathcal{V} \otimes \mathcal{F}_1, \quad C_1(\mathcal{V}) = \frac{\mathcal{V}^{\otimes 2} \otimes \mathcal{F}_2}{J_1}, \quad C_2(\mathcal{V}) = \frac{\mathcal{V}^{\otimes 3} \otimes \mathcal{F}_3}{J_2}$$

For $f(z_1, z_2) = f(z_1 - z_2) \in \mathcal{F}_2$ we put

$$d_1(a \otimes b \otimes f(z_1, z_2)) = \mathrm{Res}_{z=0} f(z) Y(a, z) b = \sum f_n a_{(n)} b,$$

Here $f(z) = \sum_n f_n z^n$.

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

Given $f(z_1, z_2, z_3) \in \mathcal{F}_3$, we consider the Laurent series expansions

$$f(z_1, z_2, z_3) = \sum_k f_{12,k}(z_2, z_3)(z_1 - z_2)^k$$

$$= \sum_k f_{13,k}(z_2, z_3)(z_1 - z_3)^k = \sum_k f_{23,k}(z_1, z_3)(z_2 - z_3)^k$$

Then

$$d_2(a \otimes b \otimes c \otimes f(z_1, z_2, z_3)) = \sum_k a \otimes b_{(k)}c \otimes f_{23,k}(z_1, z_2)$$

$$- \sum_k b \otimes a_{(k)}c \otimes f_{13,k}(z_1, z_2) - \sum_k a_{(k)}b \otimes c \otimes f_{12,k}(z_1, z_2).$$

The following identity follows from the Borcherds identity

$$d_1 \circ d_2 = 0.$$

Denote the homology of above complex by $H^{ch}_{\bullet}(\mathcal{V})$.

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Goal

BV quantization and "Chiral trace map"

$$\mathrm{Tr}^{ch}(-) : H^{ch}_{\bullet}(\mathcal{V}) \to \mathbb{C}((\hbar)).$$

$\Longrightarrow$ 2d chiral analogue of algebraic index theory.

BV formalism: 2d Chiral CFT

BV quantization of 2d chiral CFT (for example, $\beta\gamma - bc$ system).

- Local observable algebra : $\mathrm{Obs}_{2d} = \mathcal{V}^{\beta\gamma - bc}(\mathbf{h})$

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We want to construct a $\mathbb{C}[\![\hbar]\!]$-linear map

$$\langle - \rangle_{2d} : C_\bullet(\mathcal{V}^{\beta\gamma-bc}(\mathbf{h})) \overset{\text{“?”}}{\to} \mathcal{A}_{2d}(\!(\hbar)\!)$$

satisfying

$$\text{QME}: \qquad (d_{ch} + \hbar\Delta)\langle - \rangle_{2d} = 0$$

Chiral conformal block

Correlation function of local observables in a chiral CFT on a Riemann surface Σ

$$\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle , \qquad \mathcal{O}_i \in \mathcal{V}$$

is given by chiral conformal blocks on Σ.

It produces functions/forms on $\mathrm{Conf}_n(\Sigma)$ with meromorphic poles of possibly arbitrary order along the diagonals

$$\Delta = \bigcup_{1 \le i \ne j \le n} \Delta_{ij}, \quad \Delta_{ij} := \{(z_1, \cdots, z_n) \in \Sigma^n | z_i = z_j\}.$$

Quantum master equation

Similar to TQM, the 2d chiral analogue $\langle - \rangle_{2d}$ of a solution of QME is given by the following integral

$$"\stackrel{?}{=}" \int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle.$$

Here the $\mathcal{O}_i$'s are 2-form valued operators on Σ.

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\mathrm{Conf}_n(\Sigma)$.

We need to give a precise meaning to the naively divergent integral

$$\int_{\Sigma^n} \Omega$$

where Ω is a differential form on the product Σ^n with arbitrary meromorphic poles along the diagonals.

Regularized integral (L-Zhou 2020)

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

where α is a 2-form with at most logarithmic pole along D, β is a $(0,1)$-form with arbitrary order of poles along D, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham. We define the regularized integral

$$\boxed{\fint_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition.

$\fint_\Sigma$ is invariant under conformal transformations. The conformal geometry of Σ gives an intrinsic regularization of the integral $\int_\Sigma \omega$.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$\fint_\Sigma \partial(-) = \int_{\partial\Sigma} (-).$$

The $\bar\partial$-operator intertwines the residue

$$\fint_\Sigma \bar\partial(-) = \mathrm{Res}(-).$$

In general, we can define

$$\fint_{\Sigma^n} (-) := \fint_{\Sigma} \fint_{\Sigma} \cdots \fint_{\Sigma} (-) \, .$$

This gives a rigorous and intrinsic definition of

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle .$$

It exhibits all the required properties:

- Holomorphic Anomaly Equation. (**L-Zhou**, in preparation)
- Contact equations. (**Gui-L-Tang**, in preparation)
- ...

Theorem (Gui-L)

For all $n \geq 0$, we can construct a map (via Feynman diagrams)

$$[-] : (V^{\beta\gamma - bc}(\mathbf{h}))^{\otimes n} \otimes \mathcal{F}_n \to \mathcal{A}_{2d}(\!(\hbar)\!) \otimes \mathcal{A}^{\bullet,\bullet}(E_\tau^n, \star\Delta).$$

Composing with the regularized integral, we get

$$\langle - \rangle_{2d} := \int_{E_\tau^n} [-] : (V^{\beta\gamma - bc}(\mathbf{h}))^{\otimes n} \otimes \mathcal{F}_n \to \mathcal{A}_{2d}(\!(\hbar)\!)$$

which is well defined on $C_\bullet(V^{\beta\gamma - bc}(\mathbf{h})), \bullet = 0, 1, 2$ and satisfying

$$QME : \quad (d_{ch} + \hbar\Delta)\langle - \rangle_{2d} = 0.$$

Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}} \right\rangle_{2d}.$$

If we quantize the theory on elliptic curve $\Sigma = E_\tau$, we expect

$$\boxed{\lim_{\bar\tau\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau} \mathcal{L}} \right\rangle_{2d} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} e^{\frac{1}{\hbar}\oint dz\mathcal{L}}, \quad q = e^{2\pi i\tau}}$$

where the operation $\lim_{\bar\tau\to 0}$ sends almost holomorphic modular forms to quasi-modular forms. $L_0 =$ conformal weight operator, $c =$ central charge. $\mathrm{Tr}_{\mathcal{H}}(-) =$$q$-character on a VOA module $\mathcal{H}$.

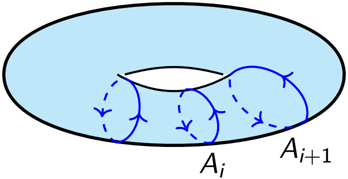
This can be viewed as an chiral algebraic index on the loop space.

Theorem (L-Zhou 2020)

Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbf{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be n disjoint A-cycles on E_τ. Then the regularized integral

$$\fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \Phi(z_1, \cdots, z_n; \tau) \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}\left[\frac{1}{\operatorname{im}\tau}\right] \quad \textit{and}$$

$$\boxed{\lim_{\bar\tau \to \infty} \fint_{E_\tau^n} \left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right) \Phi(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau)}.$$



In particular, $\fint_{E_\tau^n}$ gives a geometric modular completion for quasi-modular forms arising from A-cycle integrals.

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
n-point correlator: $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} =$integrals on the compactified configuration spaces of S^1	n-point correlator: $\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} =$regularized integrals of singular forms on Σ^n
Algebraic Index theory	Elliptic Chiral Algebraic Index

Joint work in progress with **Zhengping Gui**.

Application: Higher genus mirror symmetry

Quantum B-model on general Calabi-Yau is formulated by **Costello-L** (2012) generalizing **Bershadsky-Cecotti-Ooguri-Vafa**'s Kodaira-Spencer gravity on Calabi-Yau 3-folds. We call it

quantum BCOV theory.

It is conjectured to be mirror to higher genus Gromov-Witten invariants in certain holomorphic limit.

Quantum BCOV theory on elliptic curves is solved (**L** 2016) by the chiral deformation of free chiral boson

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z \phi)}{k+2}$$

where

$$W^{(k)}(\partial_z\phi) = (\partial_z\phi)^k + O(\hbar)$$

are the bosonic realization of the $W_{1+\infty}$-algebra.

Higher genus mirror symmetry on elliptic curves

- The chiral index of quantum BCOV theory is

$$\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar} \sum_{k \geq 0} \oint_A \eta_k \frac{W^{(k+2)}}{k+2}}$$

- The chiral index coincides with the stationary Gromov-Witten invariants on the mirror elliptic curve computed by **Dijkgraaf** and **Okounkov-Pandharipande**.

$$\boxed{\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \langle \mathrm{Stationary} \rangle_E^{\mathrm{GW}}}$$

In this case, we find [**L**, 2016[]

Quantum Mirror Symmetry=Boson-Fermion Correspondence.

Thank you!