Heterotic σ-models via BV quantization and formal geometry

Joint work with Owen Gwilliam, James Ladouce, and Brian Williams

Matt Szczesny

Boston University

Goal:

- Give a mathematically rigorous quantization of a twisted heterotic $(0,2)$ σ-model with target a complex manifold X with holomorphic gauge bundle $E \to X$ in perturbation theory.
- Relate this quantization to a sheaf of vertex algebras on the target X.
- Our tools: BV formalism (as mathematically framed by K. Costello), formal geometry (Gelfand-Kazdhan), factorization algebras (Costello-Gwilliam)

Outline:

- Physics story (Caveat Emptor !)
- Mathematical prehistory
- BV quantization
- Formal geometry
- Formulating the model, quantizing, and anomalies

Physics story:

We consider a model of maps $\Phi : \Sigma \to X$ from a Riemann surface Σ to a complex Hermitian manifold X equipped with a holomorphic vector bundle $E \to X$, with action:

$$S = \int_\Sigma |d^2z| \frac{1}{2} g_{i\bar{j}} (\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + g_{i\bar{j}} \psi^i D_z \psi^{\bar{j}} + \lambda_a D_{\bar{z}} \lambda^a$$
$$+ F^a_{bi\bar{j}}(\phi) \lambda_a \lambda^b \psi^i \psi^{\bar{j}} - l_a l^a,$$

- left-moving fermions - λ^a and λ_a are sections $K^{1/2} \otimes \Phi^* \mathcal{E}$ and $K^{1/2} \otimes \Phi^* E^*$
- right-moving fermions - ψ^i and $\psi^{\bar{j}}$ are sections of $\overline{K}^{1/2} \otimes \Phi^* TX$ and $\overline{K}^{1/2} \otimes \Phi^* \overline{TX}$
- l^a and l_a are sections of $K^{1/2} \otimes \overline{K}^{1/2} \otimes \Phi^* E$ and $K^{1/2} \otimes \overline{K}^{1/2} \otimes \Phi^* E^*$

This model has $(0,2)$ supersymmetry, allowing it to be *twisted*. The new model has action

$$S_{twist} = \int_\Sigma |d^2 z| \frac{1}{2} g_{i\bar{j}} (\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + g_{i\bar{j}} \psi^i_{\bar{z}} D_z \psi^{\bar{j}} + \lambda_{za} D_{\bar{z}} \lambda^a + F^a{}_{bi\bar{j}}(\phi) \lambda_{za} \lambda^b \psi^i_{\bar{z}} \psi^{\bar{j}} - l_{za} l^a_{\bar{z}}$$

$$\begin{aligned} \lambda^a &\in \Gamma\left(\Phi^* \mathcal{E}\right), & \lambda_{za} &\in \Gamma\left(K \otimes \Phi^* \mathcal{E}^*\right), \\ \psi^i_{\bar{z}} &\in \Gamma\left(\overline{K} \otimes \Phi^* TX\right), & \psi^{\bar{i}} &\in \Gamma\left(\Phi^* \overline{TX}\right), \\ l^a_{\bar{z}} &\in \Gamma\left(\overline{K} \otimes \Phi^* \mathcal{E}\right), & l_{za} &\in \Gamma\left(K \otimes \Phi^* \mathcal{E}^*\right). \end{aligned} \tag{1}$$

The twisted model has one remaining right-moving nilpotent supersymmetry $\bar{Q}_+$, $\bar{Q}_+^2 = 0$.

- We are interested in this model in perturbation theory, when only maps $\Phi : \Sigma \to X$ near constant maps contribute - no instanton contributions.
- Physical arguments (Witten, Kapustin, Nekrasov, Meng-Chwan Tan) show that in perturbation theory the $\bar{Q}_+$-invariant observables are locally described by a $\beta\gamma - bc$ system. In mathematical terms, we have a sheaf of observables on X whose sections look locally like a $\beta\gamma - bc$ chiral/vertex algebra. I will denote this sheaf by $CDO(X, E)$

- The global (on X) problem of describing the perturbative observables can therefore be formulated in terms of gluing $\beta\gamma - bc$ systems on coordinate patches.
- There are anomalies associated with consistent gluing: $ch_2(TX) - ch_2(E) = 0$. Existence of stress tensor obstructed by $c_1(TX) - c_1(E)$.

Mathematical prehistory:

The sheaf $CDO(X, E)$ appeared in the mathematics literature in the work of Malikov-Schechtman-Vaintrob-Gorbounov (1998-2000) under the name of *chiral differential operators*:

Theorem (MSV, GMS)

Let X be a complex manifold of dimensions n equipped with a holomorphic vector bundle $E \to X$ of rank k. If $ch_2(TX) - ch_2(E) = 0$, then X carries a sheaf of vertex algebras $CDO(X, E)$ having the property that $H^0(CDO(\mathbb{C}^n, E))$ is the $\beta\gamma - bc$ system of rank (n, k). If $c_1(TX) - c_1(E) = 0$, this is a sheaf of conformal vertex algebras.

Vertex algebras

A *vertex algebra* is a $\mathbb{Z}_+$-graded vector space $V = \oplus_n V_n$ equipped with a state-field correspondence (or vertex operator)

$$Y : V \to End(V)[[z, z^{-1}]]$$

$$A \to Y(A, z) = A_n z^{-n-1}$$

satisfying a number of properties. Some of these are:

- There is a *vacuum* vector $|vac> \in V_0$ such that $Y(|vac>, z) = Id_V$.
- (locality) for any two $A, B \in V$, $(z-w)^N [Y(A,z), Y(B,w)] = 0$ for N sufficiently large.

As a consequence of the vertex algebra axioms, for any two $A, B \in V$, we have an *operator product expansion* (OPE):

$$Y(A,z)Y(B,w) = \frac{C_N(w)}{(z-w)^N} + \cdots + \frac{C_1(w)}{(z-w)^1} + \text{reg. at } z = w$$

We write

$$Y(A,z)Y(B,w) \sim \frac{C_N(w)}{(z-w)^N} + \cdots + \frac{C_1(w)}{(z-w)^1}$$

Commutators $[A_n, B_m]$ can be computed from the OPE via residues.

V is said to be *conformal* if there is a vector $\omega \in V$ such that $T(z) = Y(\omega, z)$ has OPE

$$T(z)T(w) \sim \frac{C/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}$$

which is equivalent to the coefficients of $T(z) = \sum_n L_n z^{-n-2}$ satisfying the commutation relations of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}C$$

where $C \in \mathbb{C}$ is called the *central charge*.

$\beta\gamma - bc$ system of rank (n, k)

Start with Lie superalgebra with generators $\beta^i_r, \gamma^i_r, b^j_r, c^j_r$, where $r \in \mathbb{Z}$, $i = 1 \cdots n$, $j = 1 \cdots k$.

$$[\beta^i_r, \gamma^{i'}_s] = \delta_{ii'}\delta_{r,-s}, \quad [b^j_r, c^{j'}_s]_+ = \delta_{jj'}\delta_{r,-s}$$

- The vertex algebra $\beta\gamma bc(n, k)$ has underlying vector space a Fock module for this Lie algebra.
- It is generated by fields $\beta^i(z), \gamma^i(z), b^j(z), c^j(z)$ with OPE's

$$\beta^i(z)\gamma^{i'}(w) \sim \frac{\delta_{ii'}}{z-w} \quad b^j(z)c^{j'}(w) \sim \frac{\delta_{jj'}}{z-w}$$

The construction of MSV, GMS amounts to constructing an action of coordinate changes and gauge transformations on $\beta\gamma bc(n, k)$.

- The anomaly $ch_2(TX) - ch_2(E)$ arises because only a central extension of this group acts in general.
- The anomaly $c_1(TX) - c_1(E)$ arises as an obstruction to defining the section corresponding to ω - i.e. $T(z)$ globally.

It was not clear from original construction of MSV and GMS how the sheaves $CDO(X, E)$ are related to physics - i.e. how they arise as a solution to a quantization problem.

Goal:

- To obtain $CDO(X, E)$ by a mathematically rigorous BV quantization of a simple model.
- To exhibit the $ch_2(TX) - ch_2(E)$ and $c_1(TX) - c_1(E)$ as obstructions to quantization (consistent with certain symmetries).
- To explicitly see the modular properties of the partition function in geometric terms.

Earlier work:

- Kevin Costello - *A geometric construction of the Witten genus I/II* - Proceedings of ICM 2010/ arXiv:1112.0816
- Vassily Gorbounov, Owen Gwilliam, and Brian Williams - *Chiral differential operators via quantization of the holomorphic σ-model.* - Asterisque 2020. - deals with the case $E = 0$.

A sketch of the BV formalism following K. Costello

Roughly speaking, the BV formalism is a tool for quantizing a field theory consistently with the symmetry of some Lie algebra $\mathfrak{g}$.

Setup: we have

- a space of fields $\mathcal{F}$
- an action functional $S : \mathcal{F} \to \mathbb{R}$
- an action of $\mathfrak{g}$ on $\mathcal{O}_{loc}(\mathcal{F})$ - the local functionals on $\mathcal{F}$ leaving S invariant.

In this situation we can embed the problem in $\mathcal{F}^{BV} = T^*[-1](\mathfrak{g} \oplus \mathcal{F})$. This shifted cotangent bundle carries a shifted symplectic pairing of degree -1

- Functionals on $\mathcal{F}^{BV}$ carry a shifted Poisson bracket $\{,\}$ of degree $+1$.
- $S_{BV} = S + h_X$ satisfies the BV Classical Master Equation $\{S_{BV}, S_{BV}\} = 0$

Perturbative BV quantization involves deforming S_{BV} to an element of $\mathcal{O}_{loc}(\mathcal{F}^{BV})[[h]]$ satisfying the BV Quantum Master Equation:

$$QI + 1/2\{I, I\} + h\Delta I = 0$$

where we have written $S_{BV}(\phi) = \langle \phi, Q\phi \rangle + I_{BV}(\phi)$ (quadratic + interaction piece) where

- Δ is the BV Laplacian - a type of divergence operator.
- QME says symmetry preserves "path integral measure" on $\mathcal{F}^{BV}$..
- Making sense of QME and Δ in perturbation theory requires Feynman diagrams and regularization/renormalization. K. Costello makes sense of this via effective field theory approach.
- Anomalies (obstructions to solving QME) appear as certain Feynman diagrams. These can in turn sometimes be identified with local incarnations of characteristic classes.

Question: what's gained by embedding $\mathfrak{g}[1] \oplus \mathcal{F}$ in $\mathcal{F}^{BV}$?
Answer: The freedom to replace integral over $\mathcal{F}$ with integrals over other Lagrangian submanifolds $\mathcal{F}^{BV}$ (these choices correspond to choices of gauge-fixing conditions).

Formal geometry - (Gelfand-Kazhdan

This is a version of the associated bundle construction. Recall: If G is a group, X a manifold, $P \to X$ a principal G-bundle, and V a representation of G, we can form the associated bundle

$$P \times_G V$$

.

Suppose X is a complex manifold of dimension n. Then at each $x \in X$, $\hat{\mathcal{O}}_x \simeq \mathbb{C}[[z_1, \cdots, z_n]]$. A choice of a such an isomorphism is called a *formal coordinate system* at x.

Let $\hat{X} = \{x \in X, (z_i)_{i=1}^n\}$, where $(z_i)_1^n$ is a formal coordinate system at x.

- $\hat{X} \to X$ has the structure of a principal bundle for the pro-algebraic group Aut_n of automorphisms of $\mathbb{C}[[z_1, \cdots, z_n]]$ preserving the origin.
- Any representation V of Aut_n yields an associated bundle over X.
- Example: If $V = \mathbb{C}[[z_1, \cdots, z_n]]$,

$$\hat{X} \times_{Aut_n} V \simeq J(\mathcal{O}),$$

where $J(\mathcal{O})$ is the jet bundle of $\mathcal{O}_X$.

Punchline:

Any construction on $\hat{D}_n = Spec(\mathbb{C}[[z_1, \cdots, z_n]])$ which carries an action of Aut_n globalizes to all of X.

Can also incorporate bundles into the story. If $E \to X$ is a vector bundle of rank k, at each point $x \in X$ we have the space of *formal frames* $(e_j)_{j=1}^k$. Combining formal coordinates and formal frames we obtain a principal bundle

$$\widehat{X(E)} = \{x \in X, (z_i)_{i=1}^n, (e_j)_{j=1}^k\} \to X$$

for the (pro) group $Aut_n \ltimes GL_k(\mathbb{C}[[z_1, \cdots, z_n]])$.

Approach: to quantize a σ-model with target X, it suffices to start with a σ-model with target $\hat{D}_n$ (possibly equipped with a vector bundle E), and quantize equivariantly with respect to $Aut_n \ltimes GL_k(\mathbb{C}[[z_1, \cdots, z_n]])$.

Actually, rather than working with the group $Aut_n \ltimes GL_k(\mathbb{C}[[z_1, \cdots, z_n]])$, it suffices to work with its Lie algebra $W_n \ltimes \mathfrak{gl}_k(\mathcal{O}_{\hat{D}_n})$ keeping track of a (mild) integrability condition (language of Harish-Chandra pairs).

Heterotic sigma-model with target $\hat{D}_n$

- Fix a source Riemann surface Σ
- Let $\mathfrak{g}_n = \mathbb{C}^n[-1]$, $V_E = \mathbb{C}^k$, which we view as abelian Lie algebras.
- Note that taking Chevalley cochains $C^*_{Lie}(\mathfrak{g}_n) = \widehat{Sym}(\mathfrak{g}_n^\vee[-1]) \simeq \mathbb{C}[[z_1, \cdots, z_n]] = \mathcal{O}_{\hat{D}_n}$. Similarly $C^*_{Lie}(V_E) = \wedge^* V_E^\vee$.
- $$\begin{aligned}\mathfrak{g}_E^\Sigma =& \Omega^{0,*}(\Sigma, \mathfrak{g}_n) \oplus \Omega^{1,*}(\Sigma, \mathfrak{g}_n^\vee[-2]) \oplus \\ & \Omega^{0,*}(\Sigma, V_E) \oplus \Omega^{1,*}(\Sigma, V_E^\vee[-2]).\end{aligned}$$
- $\overline{\partial}$ equips $\mathfrak{g}_E^\Sigma$ with the structure of an abelian dgla

- $\mathfrak{g}_E^{\Sigma}[1]$ is equipped with a pairing $\langle , \rangle$ arising from pairings between $\mathfrak{g}_n$, V_E and their duals followed by integration along Σ.
- We think of $\mathcal{F} = \mathfrak{g}_E^{\Sigma}[1]$ as the fields of our formal σ-model. At this stage it's just a free theory with action functional

$$S(\phi) = \int_{\Sigma} \langle \phi, \overline{\partial}\phi \rangle, \quad \phi \in \mathcal{F}$$

- The pairing induces a shifted Poisson bracket of degree 1 on $\mathcal{O}_{loc}(\mathcal{F})$ - the space of local functionals on $\mathcal{F}$.

- We want to obtain a quantization with target $(X, E \to X)$ by quantizing the formal σ-model equivariantly with respect to the Lie algebra $W_n \ltimes \mathfrak{gl}_k(\mathcal{O}_{\hat{D}_n})$.
- Passing to the BV action S_{BV} makes this theory no longer free. We can attempt to solve the QME in perturbation theory.

Theorem (GLSW)

- *The obstruction to quantizing the formal heterotic σ-model equivariantly with respect to* $W_n \ltimes \mathfrak{gl}_k(\mathcal{O}_{\hat{D}_n})$ *can be identified with* $ch_2(TX) - ch_2(E)$.
- *The observables of the the quantized theory can be identified with* $CDO(X, E)$.

Let $\mathcal{T}_\Sigma$ denote the sheaf of holomorphic vector fields on Σ. $\mathcal{T}_\Sigma$ is another symmetry of the theory. We can attempt to quantize equivariantly with respect to $\mathcal{T}$.

Theorem

If $c_1(TX) - c_1(E)$ and $ch_2(TX) - ch_2(E)$ both vanish, we can quantize equivariantly with respect to $\mathcal{T} \oplus W_n \ltimes \mathfrak{gl}_k(\mathcal{O}_{\hat{D}_n})$. In other words, $CDO(X, E)$ is a sheaf of conformal vertex algebras.

Chiral de Rham Complex

The case $E = TX$ is special, as $ch_2(TX) - ch_2(E) = 0$ automatically. The resulting sheaf is called the *chiral de Rham complex* (MSV).

- It is known that when $c_1(TX) = 0$, this sheaf of vertex algebras has $N = 2$ super-Virasoro action
- We can see this by replacing Σ by a $(1,1)$ super Riemann surface.
- Partition function of model is two-variable elliptic genus $Ell(X, q, y)$ - we can construct it directly on the universal Jacobian over M_1. Makes explicit the modularity property of $Ell(X, q, y)$ - weak Jacobi form.