# Regularized Integrals on Riemann Surfaces and Mirror Symmetry for Elliptic Curves

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# Overview

Goal:

1. Develop a new technique to "regularize" Feynman graph integrals arising from studies of chiral deformations of 2d CFTs
2. Apply the technique to the studies of mirror symmetry for elliptic curves



w/ Si Li, arxiv: 2008.07503

# Introduction and background: Gromov-Witten theory

$X$: smooth algebraic variety over $\mathbb{C}$

Gromov-Witten theory [Ruan-Tian, Li-Tian, Behrend-Fantechi,$\cdots$]

- "counting" of stable maps $\Sigma \to X$

$$N_{g,d} \text{ ``} = \text{''} \, \overline{\{u : \Sigma \to X \mid g(\Sigma) = g, \; u_*[\Sigma] = d \in H_2(X, \mathbb{Z})\}}$$

- Gromov-Witten invariants are of central importance, connecting many branches such as symplectic geometry, algebraic geometry, enumerative geometry, integrable system, number theory, etc.

# Introduction and background: Gromov-Witten theory

- For special $X$ (e.g., toric varieties), methods are developed to compute these invariants such as localization [Kontsevich, Graber-Pandharipande, $\cdots$], topological vertex, topological recursion [Eynard-Orantin], But the complexity grows rapidly and makes the computations nearly impossible as the genus goes up.

- For general $X$, even less tools exist for direct computations of these invariants.

- Surprisingly, for the special case where $X$ is an algebraic curve, closed formulae for the generating series

$$F_g(q) = \sum_d N_{g,d} q^d \in \mathbb{Q}[[q]]\,, \quad d \in H_2(X, \mathbb{Z}) \cong \mathbb{Z}$$

can be obtained [Okounkov-Pandharipande, Bloch-Okounkov] using GW-Hurwitz correspondence.

# Introduction and background: mirror symmetry

Mirror symmetry conjecture supplies effect methods to predict/compute GW invariants.

| A-model: GW theory of $X$ | B-model: Hodge-type theory of $\check{X}$ |
|---|---|
| generating series of GW invariants | period integrals |
| axioms in GW theory | algebraic and differential relations |
| $\cdots$ | $\cdots$ |

- Genus zero mirror symmetry [Candelas-de la Ossa-Green-Parkes, Givental, Lian-Liu-Yau, $\cdots$]
- Genus one mirror symmetry [Li-Zinger, Zinger, $\cdots$]
- Higher genus mirror symmetry
  $X$=toric CY: via topological recursion [Eynard-Orantin, Fang-Liu-Zong, $\cdots$]
  $X$=elliptic curve: [Dijkgraaf, Kaneko-Zagier, Bershadsky-Cecotti-Ooguri-Vafa, Li, $\cdots$]

$\cdots$

| A-model | B-model |
| --- | --- |
| $\check{X}$=elliptic curve | $X$=elliptic curve $E = \mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau)$ |
| GW invariants on $X$ | Feynman graph integrals |
| $F_g$ | $\sum_{\Gamma: g(\Gamma)=g} \frac{1}{\lvert \mathrm{Aut}\Gamma \rvert} I_\Gamma$ |

In this talk we will mainly focus on Feynman graph integrals for the elliptic curve case.

# Introduction and background: Feynman graph integrals

$\Gamma$: labelled oriented graph without self-loops, with vertices labelled by $1, 2, \cdots, n$

$V(\Gamma)$: set of vertices, $|V(\Gamma)| = n$

$E(\Gamma)$: set of edges, each edge $e \in E(\Gamma)$ connecting its head $h(e) \in V(\Gamma)$ and tail $t(e) \in V(\Gamma)$

$\mathrm{Aut}\Gamma$: automorphism group of the labelled oriented graph $\Gamma$.

# Introduction and background: Feynman graph integrals

Feynman rule

- to each labelled vertex $v \in V(\Gamma)$ is assigned a copy of elliptic curve $E$, denoted by $E_v$
- to each edge $e \in E(\Gamma)$ is assigned a "propagator" $P_e(z_{h(e)} - z_{t(e)})$ where

$$P(z-w) = \partial_z \bar{\partial}^*_{\bar{w}} G \text{ “ $=$ ” } \partial_z \frac{1}{\bar{\partial}_{\bar{w}}} = \wp(z-w) + \eta_1^*$$

  with $G$ given by the Green's function with respect to the flat metric on $X$ whose Kähler form is

$$vol = \frac{\sqrt{-1}}{2 \operatorname{im} \tau} dz \wedge d\bar{z}$$

- to each such graph $\Gamma$ is assigned a function

$$\Phi_\Gamma = \prod_{e \in E(\Gamma)} P_e(z_{h(e)} - z_{t(e)}) : \prod_{v \in V(\Gamma)} E_v \to \mathbb{C}$$

# Introduction and background: Feynman graph integrals

- Feynman graph integral

$$\Phi_\Gamma = \prod_{e \in E(\Gamma)} P_e(z_{h(e)} - z_{t(e)}) : \prod_{v \in V(\Gamma)} E_v \to \mathbb{C}$$

$$I_\Gamma \text{ ``} = \text{''} \int_{\prod_v E_v} \Phi_\Gamma \prod_v \mathrm{vol}_v$$

- Mirror of higher genus GW invariants on $\check{X}$ in terms of integrals on $X$

$$F_g = \sum_{\Gamma : g(\Gamma) = g} \frac{1}{|\mathrm{Aut}\Gamma|} I_\Gamma$$

# Introduction and background: Feynman graph integrals

$$I_\Gamma \text{ ``}=\text{''} \int_{\prod_v E_v} \Phi_\Gamma \prod_v \mathrm{vol}_v$$

The propagator

$$P(z,w) \in \mathcal{O}_{E\times E}(2\Delta)$$

has a pole of order two along the diagonal $\Delta$ given by $z = w$ in $E \times E$. The Feynman weight is meromorphic on $\prod_v E_v$

$$\Phi_\Gamma \in \mathcal{O}_{\prod_v E_v}(\Delta_\Gamma)\,, \quad \Delta_\Gamma = \sum_{e\in E(\Gamma)} 2\Delta_{h(e),t(e)}$$

where $\Delta_{h(e),t(e)}$ is the divisor in $\prod_v E_v$ given by $z_{h(e)} = z_{t(e)}$.

# Goal: regularizing Feynman graph integrals

The goal of this talk is to explain a natural regularization of the divergent Feynman graph integral [Li-Z]

$$I_\Gamma \text{``}=\text{''} \int_{\prod_v E_v} \Phi_\Gamma \prod_v \mathrm{vol}_v$$

We shall see that

- The new notion of regularized integral is developed for integrals of on differential forms on configuration spaces of Riemann surfaces, with holomorphic poles.
- It is intrinsic, has many desired properties similar to those of ordinary integrals, and works for the relative version.
- Besides the context of mirror symmetry, it also provides a tool in understanding integrals of the same type in various 2d chiral conformal field theories.

Section 2

# Regularized integrals

- Definition/constructions
- Properties
- Examples

# Regularized integrals: notations

$X$: smooth Riemann surface

- $\Delta_{ij} = \{(p_1, p_2, \cdots, p_n) \in X^n \mid p_i = p_j\} \subseteq X^n$
  $\Delta = \cup_{i \neq j} \Delta_{ij}$: big diagonal
- $\Omega^{p,q}_{X^n}(*\Delta)$: sheaf of holomorphic $(p,q)$ forms with arbitrary poles only along $\Delta$
- $\mathcal{A}^{p,q}_{X^n}(*\Delta) := \Omega^{p,q}_{X^n}(*\Delta) \otimes_{\mathcal{O}_{X^n}} \mathcal{C}^{\infty}_{X^n}$: sheaf of smooth $(p,q)$ forms with arbitrary holomorphic poles only along $\Delta$

Note: $\Omega^{p,q}_{X^n}(*\Delta)$ and $\mathcal{A}^{p,q}_{X^n}(*\Delta)$ include interesting classes of $(p,q)$ forms on the configuration space $\mathrm{Conf}_n(X) = X^n - \Delta$, which are of central importance in studying correlation functions/conformal blocks in 2d chiral CFTs.

# Regularized integrals: notations

$X$: smooth Riemann surface

- $\Omega_{X^n}^{p,q}(\log\Delta)$: sheaf of $\mathcal{O}_{X^n}$-modules generated by $\Omega_{X^n}^{p,q}(\log\Delta_{ij}), i\neq j$
  Locally, around

  $$p\in\Delta_I := \{(p_1,\cdots,p_n)\in X^n \mid p_a = p_b\,,\forall a,b\in I\}\,, I\subseteq\{1,2,\cdots,n\}$$

  for a small neighborhood $U$ containing $p$, one has

  $$\Omega_{X^n}^{p,q}(\log\Delta)(U) = \Omega_{X^n}^{p,q}(U)[\frac{dz_{ij}}{z_{ij}}]_{i,j\in I}/\sim$$

- $\mathcal{A}_{X^n}^{p,q}(\log\Delta) := \Omega_{X^n}^{p,q}(*\Delta)\otimes_{\mathcal{O}_{X^n}}\mathcal{C}_{X^n}^{\infty}$: smooth $(p,q)$ forms with possible holomorphic log-poles only along $\Delta$

Note: when $n=2$, $\Delta$ is normal crossing, and when $n=1$ with $\Delta$ replaced by a reduced effective divisor $D$, the sheaf $\Omega_{X^n}^{p,q}(\log\Delta)$ reduces to the familiar sheaf of log forms.

# Regularized integrals: definition/construction

Quick facts:

- $(\mathcal{A}^{\bullet,\bullet}_{X^n}(*\Delta), \partial, \bar{\partial})$ is a double complex
- $(\mathcal{A}^{\bullet,\bullet}_{X^n}(\log\Delta), \partial, \bar{\partial})$ is a double complex
- Elements in $\Gamma(X^n, \mathcal{A}^{n,n}_{X^n}(\log\Delta))$ are absolutely integrable on $X^n$.

## Definition (Li-Z)

Denote $A^{\bullet,\bullet}(*\Delta) = \Gamma(X^n, \mathcal{A}^{\bullet,\bullet}_{X^n}(*\Delta))\,, A^{\bullet,\bullet}(\log\Delta) = \Gamma(X^n, \mathcal{A}^{\bullet,\bullet}_{X^n}(\log\Delta))$
Then one has the "$\partial$-retraction" data

$$\begin{tikzcd} A^{n,\bullet}(\log\Delta), \partial \arrow[r, bend left, "i"] & A^{n,\bullet}(*\Delta), \partial \arrow[l, bend left, "r"] \arrow[loop right, "h"] \end{tikzcd}$$

For any $\omega \in A^{n,n}(*\Delta)$, one defines $⨍_{X^n}\omega := \int_{X^n} r(\omega)$.

# Regularized integrals: definition/construction

- The existence of the $\partial$-retraction is proved by using "reduction of pole" coherently. Concretely, for any $\omega \in A^{n,\bullet}(*\Delta)$, there exists a decomposition

$$\omega = \alpha + \partial\beta\,, \quad \alpha \in A^{n,\bullet}(\log\Delta)\,, \beta \in A^{n-1,\bullet}(*\Delta)$$

  Then

$$⨍_{X^n} \omega := \int_{X^n} r(\omega) = \int_{X^n} \alpha$$

- Independence of choice of $\alpha, \beta$ follows from the Stokes theorem and the crucial observation

$$\mathrm{Res}_{p\in D}\beta = 0\,, \quad \forall \beta \in A^{0,1}(*D)$$

  and induction (Fubini theorem). The above observation is no longer true had we chosen to use $\bar{\partial}$-retraction (which does not not exist anyway) $\omega = \alpha + \bar{\partial}\beta$.

# Regularized integrals: properties

**Theorem (Li-Z)**

*This notion of regularized integrals satisfies many nice properties:*

- *works for Riemann surfaces with boundary*
- *Fubini theorem, Stokes theorem, Riemann-Hodge bilinear relation,* $\cdots$
- *when $n = 1$, it reduces to the Cauchy principal value*

$$\fint_X \omega = \lim_{\epsilon \to 0} \int_{X - \cup_{i=1}^m B_\epsilon(p_i)} \omega\,, \quad \omega \in A^{1,1}(*\sum_{i=1}^m p_i)$$

- *works for families of Riemann surfaces (i.e., relative version)*

# Regularized integrals: examples

## Example

elliptic curve: $X = E = \mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau)$, $O = [0]$
$\omega = \wp(z)\mathrm{vol}$, where

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}\right) \in H^0(X, \mathcal{O}_X(2O))$$

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m\tau+n)} - \frac{1}{(m\tau+n)} + \frac{z}{(m\tau+n)^2}\right)$$

Then

$$\wp(z) + \eta_1 + \frac{-\pi}{\operatorname{im}\tau} = -\partial_z\left(\zeta - z\eta_1 + \frac{-\pi}{\operatorname{im}\tau}(\bar{z} - z)\right)$$

# Regularized integrals: examples

## Example

$(\cdots$ continued)

$$P(z) := \wp(z) + \underbrace{\eta_1 + \frac{-\pi}{\operatorname{im}\tau}}_{\eta_1^*} \in H^0(X, \mathcal{O}_X(2O))$$

$$Z(z,\bar{z}) := \zeta - z\eta_1 + \frac{-\pi}{\operatorname{im}\tau}(\bar{z} - z) \in \Gamma(X, \mathcal{A}_X^{0,0}(O))$$

$$-\partial_z Z = P(z)$$

$$P(z)\mathrm{vol} = -\partial Z(z) \wedge (\frac{i}{2\operatorname{im}\tau} d\bar{z}) = 0 - \partial(Z(z) \wedge \frac{i}{2\operatorname{im}\tau} d\bar{z})$$

It follows that

$$\fint_X \wp(z)\mathrm{vol} = \fint_X P(z)\mathrm{vol} - \fint_X \eta_1^* \mathrm{vol} = -\eta_1^*$$

# Regularized integrals: examples

## Example

($\cdots$ continued)
Checking the relation to Cauchy principal value:

$$\fint_X \wp(z)\mathrm{vol} \stackrel{?}{=} \lim_{\epsilon\to 0}\int_{X-B_\epsilon(O)} \wp(z)\mathrm{vol}$$

Lifting to the universal cover, working within the fundamental domain $F$.
Following the proof of the Riemann-Hodge bilinear relation, one has

$$\lim_{\epsilon\to 0}\int_{X-B_\epsilon(O)} \wp(z)\mathrm{vol} = \lim_{\epsilon\to 0}\int_{F-B_\epsilon(O)} \wp(z)dz\wedge d\frac{\bar{z}-z}{\bar{\tau}-\tau}$$

$$= -\lim_{\epsilon\to 0}\int_{F-B_\epsilon(O)} d(\wp(z)dz\wedge \frac{\bar{z}-z}{\bar{\tau}-\tau})$$

$$= -\int_{\partial F} \wp(z)dz\wedge \frac{\bar{z}-z}{\bar{\tau}-\tau} + 2\pi i\cdot \mathrm{Res}_O(\wp(z)dz\wedge \frac{\bar{z}-z}{\bar{\tau}-\tau})$$

# Regularized integrals: examples

**Example**

($\cdots$ continued)

$$\lim_{\epsilon\to 0}\int_{X-B_\epsilon(O)}\wp(z)\mathrm{vol}$$

$$=-\int_{\partial F}\wp(z)dz\wedge\frac{\bar{z}-z}{\bar{\tau}-\tau}+2\pi i\cdot\mathrm{Res}_O(\wp(z)dz\wedge\frac{\bar{z}-z}{\bar{\tau}-\tau})$$

$$=\int_A\wp(z)dz-2\pi i\frac{1}{\bar{\tau}-\tau}$$

$$=-\eta_1^*=\lim_{\epsilon\to 0}\int_{X-B_\epsilon(O)}\wp(z)\mathrm{vol}$$

# Regularized integrals: integrals on configuration space of elliptic curves

## Theorem (Li-Z)

*Consider the universal family of smooth elliptic curves $\mathcal{E} \to \mathcal{M}_{1,1}$. Let $\Phi \in \Gamma(\mathcal{E}^n, \mathcal{A}^{0,0}_{\mathcal{E}^n/\mathcal{M}_{1,1}}(*\Delta))$. Then*

- $\fint_{\mathcal{E}^n} \Phi \mathrm{vol}$ *is an (almost-holomorphic) modular form. That is,*

$$\fint_{\mathcal{E}^n} \Phi \mathrm{vol} \in \mathbb{C}[\eta_1^*, E_4, E_6] \subseteq \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}[\frac{1}{\mathrm{im}\,\tau}]) \cap \Gamma(\mathcal{M}_{1,1}, \mathcal{C}^{\infty}_{\mathcal{M}_{1,1}})$$

- *The holomorphic limit of* $\fint_{\mathcal{E}^n} \Phi \mathrm{vol}$ *is a quasi-modular form lying in* $\mathbb{C}[\eta_1, E_4, E_6]$*. Furthermore,*

$$\lim_{\frac{1}{\mathrm{im}\,\tau} \to 0} \fint_{\mathcal{E}^n} \Phi \mathrm{vol} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \int_{A_{\sigma(n)}} \cdots \int_{A_{\sigma(1)}} \Phi \, dz_1 \boxtimes dz_2 \cdots \boxtimes dz_n$$

*where* $\eta_1 = (\pi^2/3) E_2$*,* $E_4, E_6$ *are the usual Eisenstein series.*

# Regularized integrals: integrals on configuration space of elliptic curves

## Example (Regularized integral for relative version)

$X = E, n = 1, D = \sum_{i=1}^m p_i = O, \Phi \in \mathcal{O}_X(*D) = \wp(z)$. Then

$$\fint_X \Phi \mathrm{vol} = \lim_{\epsilon \to 0} \int_{X - B_\epsilon(O)} \Phi(z) \mathrm{vol}$$

$$= -\int_{\partial F} \Phi(z) dz \wedge \frac{\bar{z} - z}{\bar{\tau} - \tau} + 2\pi i \cdot \mathrm{Res}_D(\Phi(z) dz \wedge \frac{\bar{z} - z}{\bar{\tau} - \tau})$$

$$= \underbrace{\int_{A^+} \Phi(z) dz}_{-\eta_1}$$

$$+ \underbrace{2\pi i \sum_{p_i \in D} \frac{\bar{p}_i - p_i}{\bar{\tau} - \tau} \cdot \mathrm{Res}_{p_i}(\Phi(z) dz)}_{0} + \underbrace{2\pi i \frac{1}{\bar{\tau} - \tau} \sum_{p_i \in D} \mathrm{Res}_{p_i}(\Phi(z) dz \wedge (p_i - z))}_{\pi / \mathrm{im}\, \tau}$$

# Regularized integrals: integrals on configuration space of elliptic curves

Using Fubini theorem for regularized integrals, one can iterate the result for the $n = 1$ case to obtain

$$\fint_{E^n} \Phi = \underbrace{\int_A \cdots \int_A \Phi}_{\text{disjoint A}-\text{cycles}} + \text{residue terms}$$

The regularized integral combines the two types of contributions in a highly structured way, yielding an almost-holomorphic modular form.

This relation has rich physics content as will be explained a little later.

Section 3

# Applications

- Mirror symmetry for elliptic curves
- Holomorphic anomaly equations

# Applications: mirror symmetry for elliptic curves

The Feynman graph integral is a two-step construction

$$\begin{tikzcd} \Gamma \arrow[r, "\Phi"] \arrow[rr, bend left, "I"] & \Phi_\Gamma \arrow[r, "\fint"] & \fint \Phi_\Gamma \end{tikzcd}$$

- The Feynman weight $\Phi_\Gamma$ is obtained by "multiplying" propagators which a priori are distributions. For 2d chiral CFTs, propagators are represented by meromorphic forms, and can be honestly multiplied. E.g., for free boson on $E$, this is

$$P(z-w) \text{``}=\text{''} \partial_z \bar{\partial}_{\bar{w}}^{-1} = (\wp + \eta_1^*) dz \boxtimes dw$$

- The Feynman weight $\Phi_\Gamma$ has holomorphic poles only along the diagonal, essentially by locality.

# Applications: mirror symmetry for elliptic curves

The Feynman graph integral is a two-step construction

$$
\begin{tikzcd}
\Gamma \arrow[r, "\Phi"] \arrow[rr, bend left, "I"] & \Phi_\Gamma \arrow[r, "\fint"] & \fint \Phi_\Gamma
\end{tikzcd}
$$

## Corollary ( Li-Z)

Feynman graph integrals arising from 2d chiral CFTs on the elliptic curve are almost-holomorphic modular forms.
In particular, this holds for the BCOV theory on the elliptic curve which by [Li] is mirror to the GW theory of the mirror elliptic curve.

# Applications: holomorphic anomaly equations

Applying to Feynman graph integrals, the physics interpretation of

$$I_\Gamma = \fint_{E^n} \Phi_\Gamma = \underbrace{\int_A \cdots \int_A}_{\text{disjoint A}-\text{cycles}} \Phi_\Gamma + \text{residue terms} \in \mathcal{O}_{\mathbb{H}}[\frac{1}{\operatorname{im}\tau}] \cap \mathcal{C}^\infty_{\mathcal{M}_{1,1}}$$

is

$$\text{correlation function} = \text{regularization using "point-splitting" scheme}$$

$$+\,\text{contribution from contact term singularity}$$

Our regularized integral combines the two types of contributions in a highly structured way, yielding the correlation function which descends to the moduli space.

It also enjoys various combinatorial patterns such as holomorphic anomaly equations.

# Applications: holomorphic anomaly equations

## Theorem (Li-Z, in prepration)

*Let* $Y = -\frac{\pi}{\operatorname{im}\tau}$ *and* $S = \sum_{i\neq j} \operatorname{Res}_{z_i=z_j} \circ (z_i - z_j)$. *Then one has the following relations between operators acting on* $\Gamma(X^n, \Omega^{n,n}_{X^n}(*\Delta))$.

- [Oberdieck-Pixton $+\varepsilon$]

$$\partial_Y \underbrace{\fint \cdots \fint}_{n} = -\frac{1}{2} \cdot \underbrace{\fint \cdots \fint}_{n-1} \circ S$$

- 

$$\underbrace{\fint \cdots \fint}_{n} = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \int_{A_{\sigma(n)}} \cdots \int_{A_{\sigma(1)}} - \sum_{k=1}^{n} \frac{1}{k!} (\frac{Y}{2})^k \underbrace{\fint \cdots \fint}_{n-k} \circ S^k$$

# Applications: holomorphic anomaly equations

For an analytic function $F(T) = \sum_{k=0}^{\infty} a_k T^k$ and a non-commutative operator $P$, introduce the formal generating series

$$F(P) = \sum_{k=0}^{\infty} a_k \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(P \circ P \cdots \circ P), \quad P = \fint, \int_A, S$$

Then the expression

$$\underbrace{\fint \cdots \fint}_{n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \int_{A_{\sigma(n)}} \cdots \int_{A_{\sigma(1)}} - \sum_{k=1}^{n} \frac{1}{k!} (\frac{Y}{2})^k \underbrace{\fint \cdots \fint}_{n-k} \circ S^k$$

can be recast in the following forms which are more closely related to physics

$$e^{\frac{1}{\hbar} \fint} = e^{\frac{1}{\hbar} \int_A} + e^{\frac{1}{\hbar} \fint} \circ (1 - e^{\frac{1}{\hbar} \frac{1}{2} YS}), \quad (\partial_Y + \circ \frac{1}{\hbar} \frac{1}{2} S) e^{\frac{1}{\hbar} \fint} = 0$$

# Potential research

- The generating functions of GW/Hurwitz invariants admit beautiful formulae in terms of Jacobi theta functions [Okounkov-Pandharipande, Bloch-Okounkov]. The corresponding physics theory is a theory of fermions on elliptic curves, which is dual to BCOV theory on elliptic curves by the fermion-boson correspondence. It seems interesting to study correlation functions also for the fermionic theory using regularized integrals. Presumably, this can produce vast generalizations of identities between theta functions and modular forms such as Fay's trisecant formula.
- The notion of regularized integrals and thus correlation functions may admit a purely cohomological formulation (e.g., mixed Hodge structures).

Thank you !