All genus open-closed Crepant Transformation Conjecture for toric Calabi-Yau 3-orbifolds

Zhengyu Zong

Department of Mathematical Sciences, Tsinghua University

Based on joint works with Bohan Fang, Chiu-Chu Melissa Liu, and Song Yu

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Table of contents

Introduction of Gromov-Witten theory

GW theory: curve counting theory.

- Given $p_1 \neq p_2 \in \mathbb{R}^2$, there is a unique line $\ell \subset \mathbb{R}^2$ passing through p_1, p_2.
- Given $p_1 \neq p_2 \in \mathbb{P}^2$, there is a unique (complex projective) line $\ell \subset \mathbb{P}^2$ passing through p_1, p_2.
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- A general degree 2 homogeneous polynomials in X_0, X_1, X_2 is of the form $a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_0X_1 + a_4X_1X_2 + a_5X_0X_2$.
- The space of degree 2 nonzero homogeneous polynomials (modulo a global constant) can be identified with $\mathbb{P}^5 = \{[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]\}$.
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- The condition of passing through a given point corresponds to a hyperplane in $\mathbb{P}^5$. Since the 5 points are assumed to be in general position, the intersection of five such hyperplanes gives us a unique point.
- The points in $\mathbb{P}^5 \backslash U$ correspond to line pairs and double lines, and no such configuration can pass through 5 points, unless three of the points are collinear. $\Longrightarrow$ There is a unique smooth conic passing these 5 points.

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Using similar method, one can count plane cubics passing through 9 points, or more generally, plane curves of degree d passing through $d(d+3)/2$ points; in each case the answer is 1.

Another direction: Count degree d rational curves.

- Genus formula for nodal plane curves: $g = \frac{(d-1)(d-2)}{2} - \delta$, where δ is the number of nodes.
- Each node is a condition of codimension 1 and so we should consider the number of degree d rational curves passing through $d(d+3)/2 - \delta = 3d - 1$ points.

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Kontsevich's formula: Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane. Then the following recursive relation holds:

$$N_d + \sum_{d_1+d_2=d, d_1, d_2 \geqslant 1} \frac{(3d-4)!}{(3d_1-1)!(3d-3d_1-3)!} d_1^3 N_{d_1} N_{d_2} d_2$$

$$= \sum_{d_1+d_2=d, d_1, d_2 \geqslant 1} \frac{(3d-4)!}{(3d_1-2)!(3d-3d_1-2)!} d_1^2 N_{d_1} d_2^2 N_{d_2}$$

Initial condition: $N_1 = 1$.

Method: Use Gromov-Witten invariants: Count maps $f: C \to X$ from algebraic curve C to a certain target space X.

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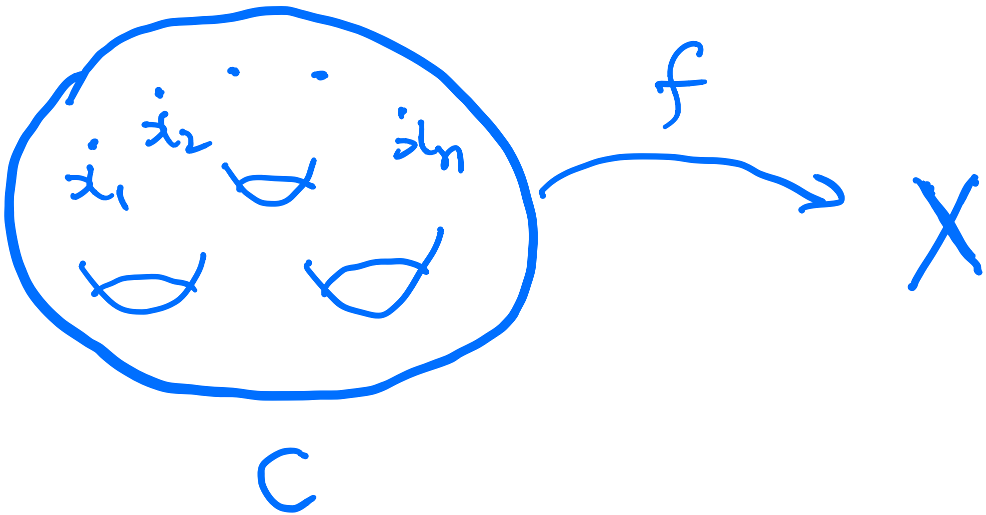


Introduction of crepant transformation

Consider a birational map $\phi : \mathcal{X}_- \dashrightarrow \mathcal{X}_+$ between two orbifolds $\mathcal{X}_\pm$. The map ϕ is called **crepant** (or $K-$**equivalent**) if there exists an orbifold $\widetilde{\mathcal{X}}$ and birational morphisms $f : \widetilde{\mathcal{X}} \to \mathcal{X}_-, g : \widetilde{\mathcal{X}} \to \mathcal{X}_-$ making the following diagram commute, such that $f^*(K_{\mathcal{X}_-}) = g^*(K_{\mathcal{X}_+})$. Here $K_{\mathcal{X}_\pm}$ is the canonical bundle of $\mathcal{X}_\pm$.

$$\begin{array}{ccccc} & & \widetilde{\mathcal{X}} & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \mathcal{X}_- & & \dashrightarrow & & \mathcal{X}_+, \end{array}$$

An example

Example

Let $\mathcal{X}_+ = K_{\mathbb{P}^2} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2), \mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$. Let

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad b_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let Σ_+ be the fan consisting of 3-dimensional cones generated by $\{b_1, b_2, b_4\}, \{b_1, b_3, b_4\}, \{b_2, b_3, b_4\}$. Let Σ_- be the fan consisting of the 3-dimensional cone generated by $\{b_1, b_2, b_3\}$. Then $\mathcal{X}_+ = X(\Sigma_+), \mathcal{X}_- = X(\Sigma_-)$

An example

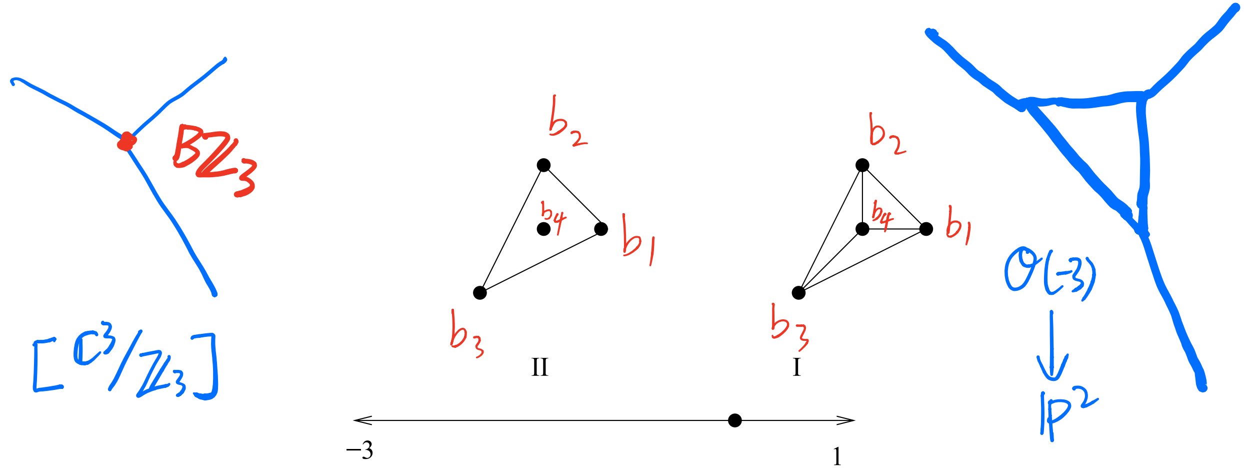
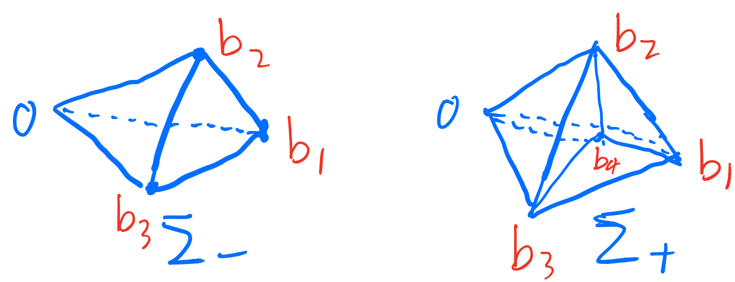


Figure: The secondary fan of $K_{\mathbb{P}^2}$



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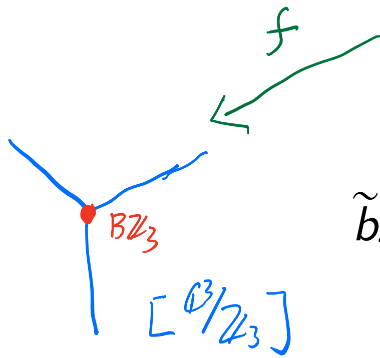

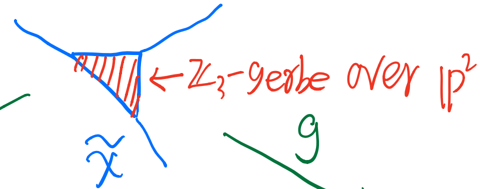

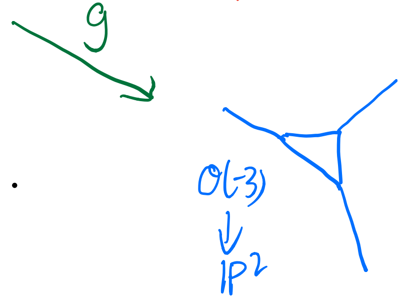

$$\widetilde{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Let $\widetilde{\Sigma}$ be the stacky fan consisting of 3-dimensional cones generated by $\{b_1, b_2, \widetilde{b}_4\}, \{b_1, b_3, \widetilde{b}_4\}, \{b_2, b_3, \widetilde{b}_4\}$. Then $\widetilde{\mathcal{X}} = X(\widetilde{\Sigma})$.

Today's goal: study the relationship between $GW(\mathcal{X}_+)$ and $GW(\mathcal{X}_-)$.

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Introduction of crepant transformation

Let $\phi : \mathcal{X}_- \dashrightarrow \mathcal{X}_+$ be a crepant transformation. Roughly speaking, the Crepant Transformation Conjecture asserts that $\mathrm{GW}(\mathcal{X}_-)$ is "**equivalent**" to $\mathrm{GW}(\mathcal{X}_+)$.

Originally introduced by **Y. Ruan**: $QH^*_{\mathrm{CR}}(\mathcal{X}_-) \cong QH^*_{\mathrm{CR}}(\mathcal{X}_+)$, $QH^*_{\mathrm{CR}}(\mathcal{X}_\pm)$ quantum cohomology.

Bryan-Graber: $F_0^{\mathcal{X}_-} = F_0^{\mathcal{X}_+}$ under change of variables and analytic continuation; generalization to higher genus Gromov-Witten potential $F_g^{\mathcal{X}_\pm}$.

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Coates-Ruan, Coates-Iritani-Tseng, Iritani: Further refinement and generalizations.

Introduction of crepant transformation

The Crepant Transformation Conjecture has been further generalized by considering higher genus Gromov-Witten invariants, open Gromov-Witten invariants, relative Gromov-Witten invariants, and so on.

Genus zero case

- $[\mathbb{C}^3/\mathbb{Z}_3]$: Bryan-Graber-Pandharipande
- $\mathbb{P}(1,3,4,4)$: Boissiere-Mann-Perroni
- **Some cases of** ***ADE*****-singularity**: Perroni
- **3-dimensional flags modulo an involution**: Gillam
- **Singular symplectic flops**: Chen-Li-Zhang-Zhao, Chen-Li-Li-Zhao
- **Ordinary flops**: Lee-Lin-Qu-Wang, Lee-Lin-Wang, Iwao-Lee-Lin-Wang
- **Toric DM stacks, complete intersections in Toric DM stacks**: Coates, Coates-Corti-Iritani-Tseng, Coates-Iritani-Tseng (examples), Coates-Iritani-Jiang (complete intersections in Toric DM stacks), Coates-Iritani-Jiang-Segal (Fourier-Mukai transformation)

Genus zero case

- $[\mathbf{Sym}^2\mathbb{P}^2]$: Wise
- **Variation of GIT quotient**: Gonzalez-Woodward
- **Birational projective Calabi-Yau manifolds**: McLean
- **Open GW of** $[\mathbb{C}^3/\mathbb{Z}_2]$: Cavalieri-Ross
- **Open GW of** $[\mathbb{C}^3/\mathbb{Z}_{n+2}], [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$: Brini-Cavalieri
- **Open GW of toric CY 3-orbifolds**: H. Ke-J. Zhou (effective brane), S. Yu (ineffective, outer-inner brane)
- ....

Higer genus case

- $[\mathbb{C}^2/\mathbb{Z}_n]$ **and** $\mathbb{C} \times [\mathbb{C}^2/\mathbb{Z}_n]$: J. Zhou
- $[\mathbb{C}^3/A_4], [\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)]$: Bryan-Gholampour
- **Some surface singularities**: X. Hu
- **Polyhedral singularity**: Bryan-Gholampour
- **Toric CY 3-orbifolds with** A_n-**singularities**: Brini-Cavalieri-Ross (open GW of $[\mathbb{C}^3/\mathbb{Z}_n]$), Ross
- **Relative GW of** $\mathbb{P}^1 \times [\mathbb{C}^2/\mathbb{Z}_n]$: Z. Zhou-Z
- $[\mathbb{C}^3/\mathbb{Z}_3]$: Coates-Iritani, Lho-Pandharipande
- **Compact toric orbifolds**: Coates-Iritani
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Today's topic

Today's topic: All genus open-closed Crepant Transformation Conjecture for toric Calabi-Yau 3-orbifolds.

Strategy: Use the Remodeling Conjecture: All genus open-closed mirror symmetry for toric CY 3-folds/3-orbifolds.

- When two toric Calabi-Yau 3-orbifolds $\mathcal{X}_-$ and $\mathcal{X}_+$ are related by a crepant transformation, they correspond to two local charts on the B-model moduli space.
- Then use the analyticity of the B-model.

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A-model

- $\mathcal{X}$ a toric CY 3-fold/3-orbifold.
- $\mathcal{L} \subset \mathcal{X}$ a certain Lagrangian (Aganagic-Vafa brane) in $\mathcal{X}$
- Consider the open Gromov-Witten potential $F_{g,n}^{(\mathcal{X},\mathcal{L})}$ of $(\mathcal{X}, \mathcal{L})$.

Open Gromov-Witten invariants: Count maps $f : C \to \mathcal{X}$, where C is a genus g bordered Riemann surface with n boundary circles such that f(boundary circles)$\subset \mathcal{L}$.

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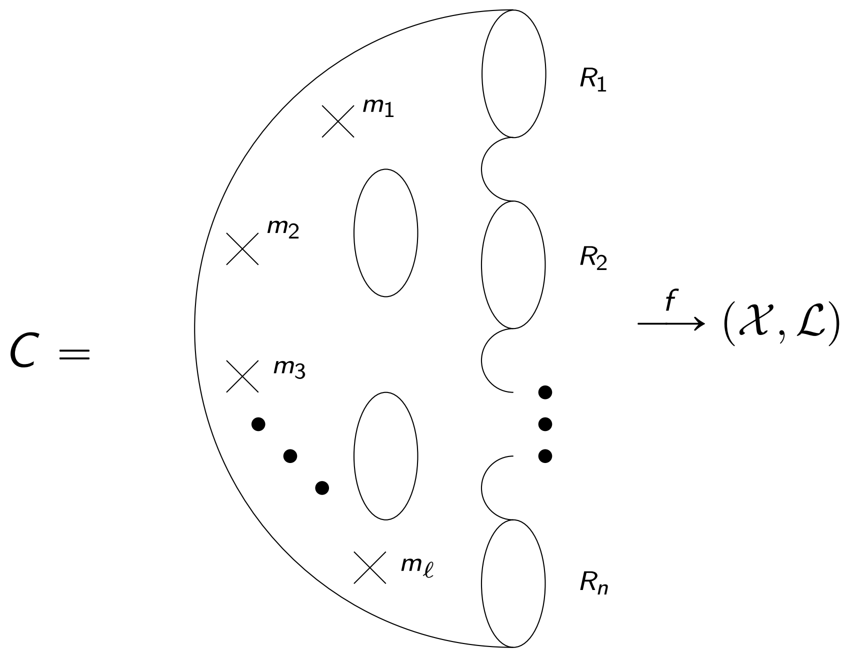

B-model

- Q: Higher genus B-model?
- Eynard-Orantin 07: Topological recursion on spectral curves $\Longrightarrow \omega_{g,n}$ symmetric n-form.
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- Let $C = \{H(X, Y) = 0\} \subset (\mathbb{C}^*)^2$ be the mirror curve of $\mathcal{X}$.
- Critical (ramification) points P_α of the map $C \to \mathbb{C}^*$ given by $(X, Y) \mapsto X$: $dX = 0$.
- $X = e^{-x}$, $Y = e^{-y}$.
- Near each ramification point P_α, use local coordinates:

$$x = x_0 + \zeta_\alpha^2, \quad y = y_0 + \sum_{i=1}^{\infty} h_i^\alpha \zeta_\alpha^i.$$

- Near each ramification point, denote $\bar{p}$:

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The initial data of the topological recursion is given by $\omega_{0,1}, \omega_{0,2}$.

- $\omega_{0,1} = ydx$.

Let the compactified mirror curve $\overline{C}$ to be of genus $\mathfrak{g}$. A_i, B_i are basis of $H_1(\bar{C};\mathbb{C})$:

- $A_i \cap B_j = \delta_{ij}$, $A_i \cap A_j = 0$, $B_i \cap B_j = 0$.

Fundamental differential of the second kind (a.k.a. Bergmann kernel) $\omega_{0,2}(p_1, p_2)$: symmetric 2-form on $\bar{C} \times \bar{C}$. It is uniquely characterized by

- $$\int_{p_1 \in A_i} \omega_{0,2}(p_1, p_2) = 0;$$
- $$\omega_{0,2}(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic} \quad p_1 \to p_2.$$

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The initial data of the topological recursion

The initial data of the topological recursion is given by $\omega_{0,1}, \omega_{0,2}$.

- $\omega_{0,1} = ydx$.

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Eynard-Orantin's topological recursion

Eynard-Orantin construct symmetric forms $\omega_{g,n}$ on C^n:

- Initial data $\omega_{0,1} = ydx$, $\omega_{0,2}$ as above;
- The recursive algorithm is:

$$\omega_{g,n+1}(p_0,\ldots,p_n)$$

$$= \sum_{P_\alpha} \mathrm{Res}_{p\to P_\alpha} \frac{\int_{\bar{p}}^{p} \omega_{0,2}(p_0,\cdot)}{2(y(p)-y(\bar{p}))dx(p)}$$

$$\cdot \Big(\omega_{g-1,n+2}(p,\bar{p},p_1,\ldots,p_n)$$

$$+ \sum_{h=0}^{g} \sum_{A\cup B=\{1,\ldots,n\},(h,|A|),(g-h,|B|)\neq(0,0)} \omega_{h,|A|+1}(p,\vec{p}_A)\omega_{g-h,|B|+1}(\bar{p},\vec{p}_B)\Big).$$

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Remodeling Conjecture

Theorem (Remodeling Conjecture)

If we expand $\omega_{g,n}$ under suitable local coordinate on the mirror curve C, we obtain the open Gromov-Witten potential $F_{g,n}^{(\mathcal{X},\mathcal{L})}$ under the mirror map.

Remodeling Conjecture

- When $\mathcal{X} = \mathbb{C}^3$, open part: L. Chen, J. Zhou; closed part: Bouchard-Catuneanu-Marchal-Sułkowski, S. Zhu.
- When $\mathcal{X}$ is smooth: Eynard-Orantin.
- General semi-projective toric CY 3-orbifolds: Fang-Liu-Z.

Global mirror curve

- $\mathcal{X}$=toric CY 3-orbifold. $\exists$ $\mathcal{C} \to \mathcal{M}_B$ *global family of mirror curves*.
- $\mathcal{M}_B$=B-model moduli space parameterizing the complex structures of the mirror curve. $\exists$ globally defined functions x, y on $\mathcal{C}$.
  $\exists$ divisor $D \subset \mathcal{M}_B$ such that for $\forall p \in \mathcal{M}_{B,0} := \mathcal{M}_B \backslash D$ the fiber $\mathcal{C}_p$ is smooth.
  $\Longrightarrow$ For $\forall p \in \mathcal{M}_{B,0}$, $(\mathcal{C}_p, x\mid_{\mathcal{C}_p}, y\mid_{\mathcal{C}_p})$ is a spectral curve.
  The mirror curve of $\mathcal{X}$ is defined over a local chart of the B-model moduli space $\mathcal{M}_B$.
- The moduli space $\mathcal{M}_B$ is constructed by the *secondary fan* of $\mathcal{X}$.

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- $\mathcal{X}_+, \mathcal{X}_-$ toric Calabi-Yau 3-orbifolds related by a crepant transformation.
  $\Longrightarrow$ The mirror curves of $\mathcal{X}_\pm$ are defined over two local charts $U_\pm$ of the B-model moduli space $\mathcal{M}_B$.
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An example

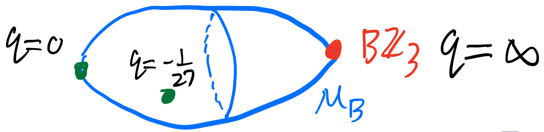
Example

$\mathcal{X}_+ = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2), \mathcal{X}_- = [\mathbb{C}^3/\mathbb{Z}_3]$. Mirror curves

$$\mathcal{X}_+ : X + Y + 1 + q_{\mathrm{LRL}} X^3 Y^{-1} = 0,$$
$$\mathcal{X}_- : q_{\mathrm{orb}} X' + Y' + 1 + X'^3 Y'^{-1} = 0.$$

They are related by an analytic continuation and a change of variables: $X = q_{\mathrm{orb}} X', Y = Y', q_{\mathrm{LRL}} = q_{\mathrm{orb}}^{-3}$.

In the family $\mathcal{C} \to \mathcal{M}_B$, we have $\mathcal{M}_B = \mathbb{P}(1,3)$ and $\mathcal{M}_{B,0} = \mathbb{P}(1,3) \setminus \{q_{\mathrm{LRL}} = 0, q_{\mathrm{LRL}} = -\frac{1}{27}, q_{\mathrm{LRL}} = \infty\}$



An example

- The point $q_{\mathrm{LRL}} = 0$ is the *large radius limit point*. The neighborhood of this point corresponds to $\mathrm{GW}(\mathcal{X}_+)$ under mirror symmetry.
- The point $q_{\mathrm{LRL}} = \infty$ is the *orbifold point*. The neighborhood of this point corresponds to $\mathrm{GW}(\mathcal{X}_-)$ under mirror symmetry.
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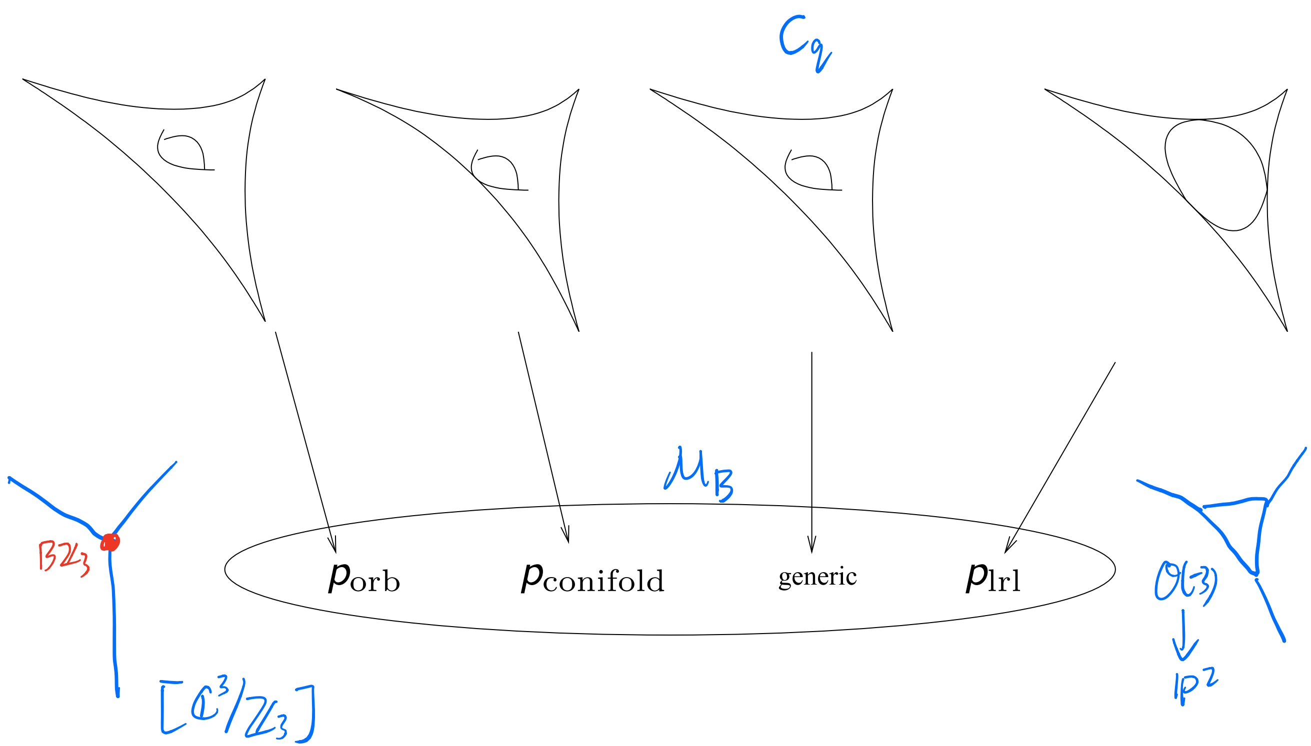
C_q
$\mathcal{M}_B$
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p_{conifold}
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Global mirror curve

- Consider the families of mirror curves C_{q_+}, C_{q_-} corresponding to $\mathcal{X}_+, \mathcal{X}_-$. Here q_+, q_- are coordinates of the two local charts U_+, U_- of $\mathcal{M}_B$.
- Let $\omega^{\pm}_{g,n}$ be the Eynard-Orantin invariants on $C_{q_\pm}$. Under mirror symmetry, $\omega^{\pm}_{g,n}$ corresponds to $F^{(\mathcal{X}_\pm, \mathcal{L}_\pm)}_{g,n}$.
- In general, $\omega^{+}_{g,n}$ and $\omega^{-}_{g,n}$ are **not** related by analytic continuation.
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Symplectic transformation

- Recall that $\omega_{0,2}(p_1, p_2)$ is uniquely determined by a choice of a system of $A-$cycles $\{A_1, \cdots, A_{\mathfrak{g}}\}$ on $\overline{C}$ such that

$$\int_{p_1 \in A_i} \omega_{0,2}(p_1, p_2) = 0;$$

- Let $\omega^{\pm}_{0,2}(p_1, p_2)$ on $\overline{C}_{q_\pm}$ be determined by $\{(A_\pm)_1, \cdots, (A_\pm)_{\mathfrak{g}}\}$. Consider the analytic continuation of $\omega^{-}_{0,2}(p_1, p_2)$ (still denoted by $\omega^{-}_{0,2}(p_1, p_2)$) to $\overline{C}_{q_+}$. Then $\omega^{-}_{0,2}(p_1, p_2)$ is determined by the parallel transport of $\{(A_-)_1, \cdots, (A_-)_{\mathfrak{g}}\}$ (still denoted by $\{(A_-)_1, \cdots, (A_-)_{\mathfrak{g}}\}$) to $\overline{C}_{q_+}$.

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Symplectic transformation

- In general, the Lagrangian subspace $L_- := \langle (A_-)_1, \cdots, (A_-)_{\mathfrak{g}} \rangle \subset H_1(\overline{C}_{q_+})$ is **different** from the Lagrangian subspace $L_+ := \langle (A_+)_1, \cdots, (A_+)_{\mathfrak{g}} \rangle \subset H_1(\overline{C}_{q_+})$.
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Symplectic transformation

- Consider a $2\mathfrak{g} \times 2\mathfrak{g}$ symplectic matrix $\mathbb{M}$

$$\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where each M_{ij} is a $\mathfrak{g} \times \mathfrak{g}$ matrix. The matrics M_{ij} is defined by

$$\begin{aligned} A_+ &= M_{11}A_- + M_{12}B_- \\ B_+ &= M_{21}A_- + M_{22}B_-, \end{aligned}$$

where

$$A_\pm = \begin{bmatrix} (A_\pm)_1 \\ \vdots \\ (A_\pm)_\mathfrak{g} \end{bmatrix}, \quad B_\pm = \begin{bmatrix} (B_\pm)_1 \\ \vdots \\ (B_\pm)_\mathfrak{g}. \end{bmatrix}$$

Symplectic transformation

- Let $(\omega_{\pm})_i, i = 1, \cdots, \mathfrak{g}$ be the holomorphic 1-form on $\overline{C}_{q_+}$ normalized by $\int_{(A_+)_j}(\omega_+)_i = \delta_{ij}, \quad \int_{(A_-)_j}(\omega_-)_i = \delta_{ij}$. We also define the column vectors

$$\omega_+ = \begin{bmatrix} (\omega_+)_1 \\ \vdots \\ (\omega_+)_{\mathfrak{g}} \end{bmatrix}, \quad \omega_- = \begin{bmatrix} (\omega_-)_1 \\ \vdots \\ (\omega_-)_{\mathfrak{g}} \end{bmatrix}$$

- Define the $\mathfrak{g} \times \mathfrak{g}$ matrix $\tau = (\tau_{ij})$ as $\tau_{ij} = \int_{(B_-)_j}(\omega_-)_i$. Let

$$\kappa := -M_{12}^T(M_{11}^T + \tau M_{12}^T)^{-1}$$

- Then we have

$$\omega_{0,2}^+(p_1, p_2) = \omega_{0,2}^-(p_1, p_2) + 2\pi\sqrt{-1}\omega_-(p_1)^T \kappa \omega_-(p_2).$$

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Graph sum formula

Applying the topological recursion, we obtain the following theorem

Theorem

The Eynard-Orantin invariants $\omega^{\pm}_{g,n}$ are related by an explicit graph sum formula which is of the form

$$\omega^{+}_{g,n} = \omega^{-}_{g,n} + \textit{graph sums of}$$
$$\{\partial^{k}_{t_i^-}\omega^{-}_{g',n'} | (g',n') < (g,n), i = 1,\cdots,\mathfrak{g}\}.$$

Here $t_1^-,\cdots,t_{\mathfrak{g}}^-$ is the coordinates of the compactly supported cohomology $H^2_{\mathrm{CR},c}(\mathcal{X}_-)$. The meaning of $(g',n') < (g,n)$ is that $g' < g$ or $g' = g, n' < n$. The graph sum formula is determined by the matrix κ which is in turn determined by the matrix $\mathbb{M}$.

Crepant Transformation Conjecture

Under mirror symmetry, the above theorem implies the Crepant Transformation Conjecture

Theorem (Crepant Transformation Conjecture)

The Gromov-Witten potentials $F_{g,n}^{(\mathcal{X}^{\pm},\mathcal{L}^{\pm})}$ are related by an explicit graph sum formula which is of the form

$$F_{g,n}^{(\mathcal{X}^{+},\mathcal{L}^{+})} = F_{g,n}^{(\mathcal{X}^{-},\mathcal{L}^{-})} + \textit{graph sums of}$$
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The graph sum formula is determined by the matrix κ which is in turn determined by the matrix $\mathbb{M}$.

Global mirror curve

For $\forall p \in \mathcal{M}_{B,0}$, consider the spectral curve $(\mathcal{C}_p, x|_{\mathcal{C}_p}, y|_{\mathcal{C}_p})$ and the compactification $\bar{\mathcal{C}}_p$. Then $\omega_{0,2}$ is determined by a choice of $A-$cycles of $\bar{\mathcal{C}}_p$.

- The choice of $A, B-$cycles has monodromies around the singular locus $\mathcal{M}_{B,\mathrm{sing}} = \mathcal{M}_B \backslash \mathcal{M}_{B,0}$ under the Gauss-Manin connection.
- $\Longrightarrow \omega_{0,2}$ is not globally defined over $\mathcal{M}_{B,0}$
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Global mirror curve

For $\forall p \in \mathcal{M}_{B,0}$, consider the spectral curve $(\mathcal{C}_p, x\mid_{\mathcal{C}_p}, y\mid_{\mathcal{C}_p})$ and the compactification $\bar{\mathcal{C}}_p$. Then $\omega_{0,2}$ is determined by a choice of $A-$cycles of $\bar{\mathcal{C}}_p$.

- The choice of $A, B-$cycles has monodromies around the singular locus $\mathcal{M}_{B,\mathrm{sing}} = \mathcal{M}_B \backslash \mathcal{M}_{B,0}$ under the Gauss-Manin connection.
- $\Longrightarrow \omega_{0,2}$ is not globally defined over $\mathcal{M}_{B,0}$
- $\Longrightarrow \omega_{g,n}$ is not globally defined over $\mathcal{M}_{B,0}$

Global mirror curve

- Solution to this problem: Introduce the *anti-holomorphic completion*.
- Fix a symplectic basis $A_1, \cdots, A_{\mathfrak{g}}, B_1, \cdots, B_{\mathfrak{g}} \in H_1(\overline{C}_p; \mathbb{Z})$. Let $\theta_1, \cdots, \theta_{\mathfrak{g}}$ be a basis of $H^0(\overline{C}_p, \Omega^1_{\overline{C}_p})$ such that $\int_{A_i} \theta_j = \delta_{ij}, \tau_{ij} = \int_{B_j} \theta_i$.

Define

- $A_i(\tau) = A_i - \sum_j (\frac{1}{\bar{\tau} - \tau})_{ij} B_j(\tau)$,
- $B_i(\tau) = B_i - \sum_j \tau_{ij} A_j$.

$\Longrightarrow A_i(\tau) \to A_i$ when $\mathrm{Im}\tau \to \infty$.

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Eynard-Orantin 07 $\Longrightarrow A_i(\tau), B_i(\tau)$ are globally defined over $\mathcal{M}_{B,0}$.

Let $\tilde{\omega}_{0,2}$ be defined by $A_1(\tau), \cdots, A_{\mathfrak{g}}(\tau)$
$\Longrightarrow \tilde{\omega}_{0,2}$ is globally defined over $\mathcal{M}_{B,0}$
$\Longrightarrow \tilde{\omega}_{g,n}$ is globally defined over $\mathcal{M}_{B,0}$ by topological recursion. $\tilde{\omega}_{g,n} \to \omega_{g,n}$ when $\mathrm{Im}\tau \to \infty$.

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Crepant resolution conjecture: 2nd formulation

Let $\mathcal{X}_+, \mathcal{X}_-$ be a two toric Calabi-Yau 3-orbifolds related by a crepant transformation.

$$\begin{array}{ccc} \tilde{\omega}_{g,n}|_{U_-} & \longrightarrow & \tilde{\omega}_{g,n}|_{U_+} \\ \text{MS}\big\downarrow \text{hol. limit} & & \text{MS}\big\downarrow \text{hol. limit} \\ F_{g,n}^{\mathcal{X}_-,\mathcal{L}_-} & \xrightarrow{\text{CTC}} & F_{g,n}^{\mathcal{X}_+,\mathcal{L}_+} \end{array}$$

The way of taking the holomorphic limit is determined by the matrix $\mathbb{M}$.

Thanks

Thank you!