# Disk Potentials for Quadrics

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# Outline



1. Motivation

2. Disk potential functions

3. Lagrangians in Quadrics

4. Disk potentials for Quadrics

5. Further directions

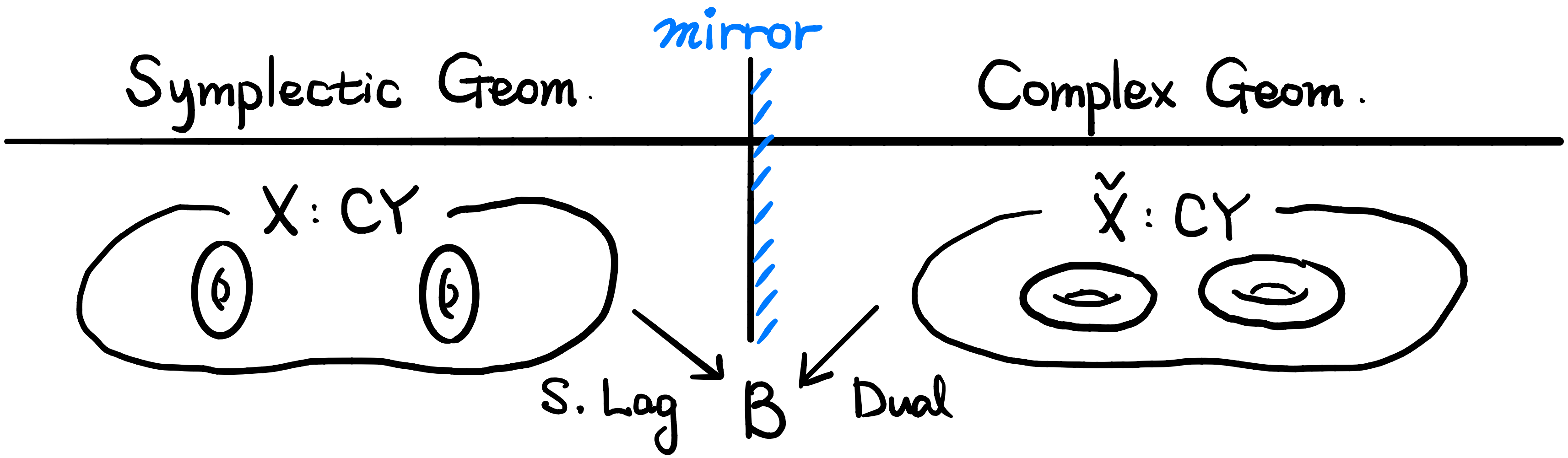
# 1. Motivation

Mirror symmetry studies duality between symplectic and complex geometry

Analogy. Fourier transform, Laplace transform

Q. How can we construct a mirror pair?

(SYZ)




Beyond CY, makes sense of mirror symmetry by replacing $\check{X}$ by $(\check{X}, W : \check{X} \to \mathbb{C})$

(Auroux) $\begin{cases} X : \text{Fano} \\ D \in H^0(X, -K_X) \end{cases} \rightsquigarrow X \setminus D$ | mirror | $(\check{X}, W)$ : Landau–Ginzburg model

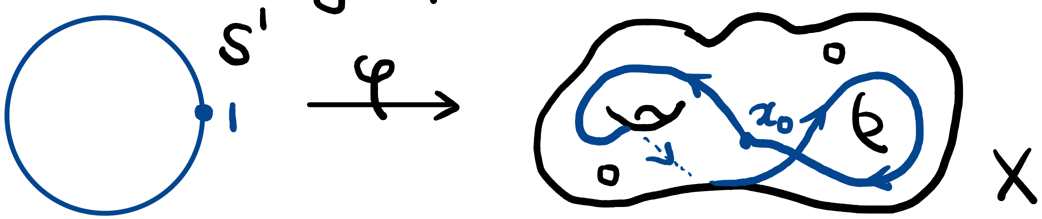
W : FOOO's disk potential

Our main interest of today's talk is $W : \check{X} \longrightarrow \mathbb{C}$ (disk potential)

## 2. Disk Potential Functions

### Idea of Gromov – Witten – Floer theory (Lagrangian Floer theory)

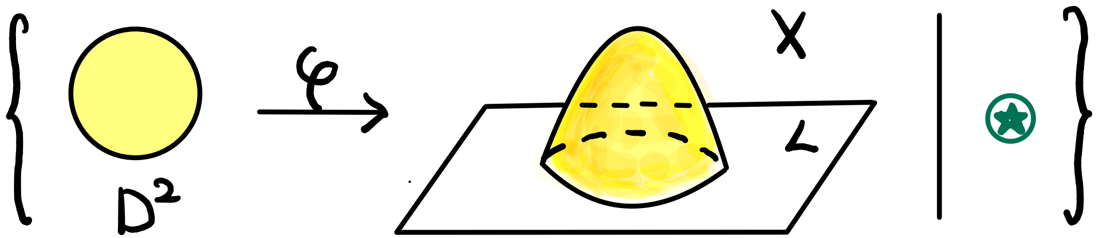
c.f> fundamental groups




$\{\varphi : (S^1, 1) \to (X, x_0)\}$ ⟵ Too gigantic to understand

$\pi_1(X, x_0) = \{\varphi : (S^1, 1) \to (X, x_0)\} / \sim$ homotopy

$(X, \omega)$ : symplectic manifold, $L$ : Lagrangian submanifold.




Choose an almost complex structure $J$ on $X$.

⊛ $\varphi$ is required to be J-holomorphic ($d\varphi \circ j = J \circ d\varphi$) + top'l constraints

$L$ : a Lagrangian torus of $(X, \omega)$

- Choose an almost complex structure $J$ on $X$.
- Fix $\beta \in \pi_2(X, L)$ (top'l constraint)

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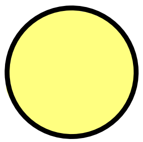
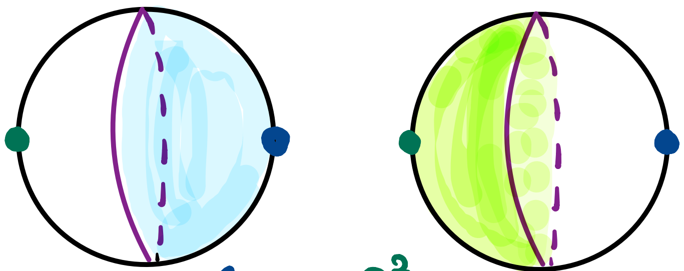
$\mathcal{M}(\beta) := \Big\{$  $\mathbb{D}^2 \xrightarrow[J\text{-holom.}]{\varphi}$  $\Big\} \Big/ \sim$



$$\left( \varphi \sim \varphi' \iff \begin{tikzcd} \mathbb{D}^2 \arrow[d] \arrow[dr] & \\ \mathbb{D}^2 \arrow[r] & (X, L) \end{tikzcd} \qquad \text{Stable map compactification} \quad xy = t \right)$$

e.g.> $(X = \mathbb{C}P^1 \simeq S^2, \omega_{FS})$

$L$ : equator

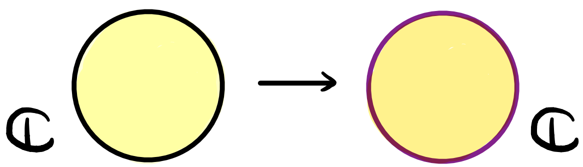




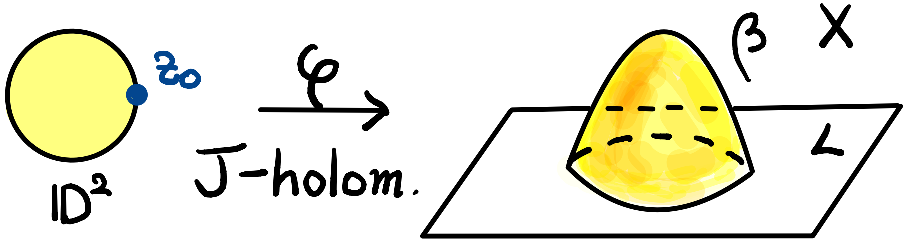
$\pi_2(X, L) \simeq \mathbb{Z}\langle \beta, S^2 - \beta \rangle$

Fix $\beta$ ($\Rightarrow$ Intersect • once)

$$\mathcal{M}(\beta) = \left\{ \varphi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a} z} : \theta \in S^1, a \in \mathring{\mathbb{D}}^2 \right\} \Big/ PSL_2\mathbb{R}$$



$$\mathcal{M}_1(\beta) := \left\{ \; \mathbb{D}^2 \ni z_0 \xrightarrow[\text{J-holom.}]{\varphi} X, \ \beta, \ L \; \right\} \Big/ \sim$$




- $ev_0 : \mathcal{M}_1(\beta) \longrightarrow L \quad (\varphi \longmapsto \varphi(z_0))$
- $\mathrm{vir.dim}_{\mathbb{R}} \mathcal{M}_1(\beta) = \underset{\theta}{\dim_{\mathbb{R}} L} + \underset{a}{\mu_L(\beta)} - 3 + 1$ e.g.) $\varphi(z) = e^{i\theta} \dfrac{z - a}{1 - \bar{a} z}$

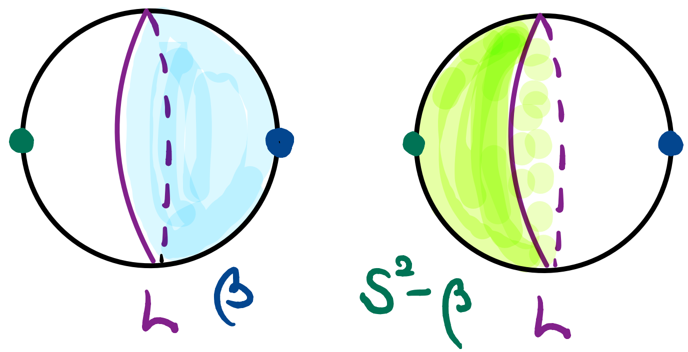
## Assumption

$\nexists$ non-constant holomorphic disks bounded by $L$ of Maslov index $\leq 0$

## Open Gromov–Witten invariants

$n_\beta :=$ the degree of $ev_0 : \mathcal{M}_1(\beta) \longrightarrow L$

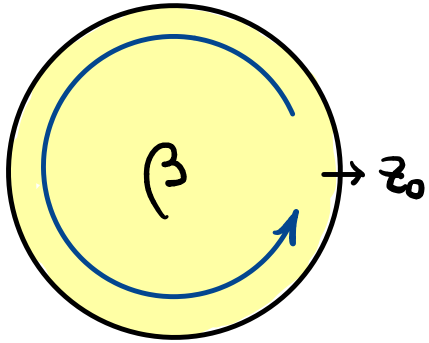
$$\left( \begin{array}{l} n_\beta \neq 0 \text{ only if } \mu_L(\beta) = 2 \\ \dim_{\mathbb{R}} \mathcal{M}_1(\beta) = \dim_{\mathbb{R}} L, \ \partial \mathcal{M}_1(\beta) = \phi \end{array} \right.$$




$n_\beta = 1$ $\quad$ $n_{S^2-\beta} = 1$

$\mathcal{L} \to L$ : trivial line bundle. $\nabla$ : a flat $\mathbb{C}^*$-connection on $\mathcal{L}$

The moduli of flat $\mathbb{C}^*$-connections serve as a mirror chart.




Take a basis

$$\pi_1(L) = \pi_1(T^n) \simeq \mathbb{Z}^n = \mathbb{Z}\langle \theta_1, \theta_2, \cdots, \theta_n \rangle \quad (\theta_i : \text{oriented loop})$$

$$\{\nabla : \text{flat } \mathbb{C}^*\text{-connections on } \mathcal{L}\}/\sim \; \simeq (\mathbb{C}^*)^n$$

$$(z_i := \mathrm{hol}_\nabla(\theta_i) \in \mathbb{C}^* \quad (i = 1, 2, \cdots, n))$$

$$\left( m_{0,\beta}(1) \xrightarrow{\nabla} m^\nabla_{0,\beta}(1) \right.$$

To define the disk potential of $L$,

- $m_{0,\beta}(1) := (ev_0)_* [\mathcal{M}_1(\beta)]$ • $m^\nabla_{0,\beta}(1) := \mathrm{hol}_\nabla(\partial\beta) \cdot m_{0,\beta}(1)$.

- $$m^\nabla_0(1) = \sum_{\beta : \mu(\beta)=2} \mathrm{hol}_\nabla(\partial\beta) \cdot m_{0,\beta}(1) \underset{\text{assumption}}{=} \sum_{\beta : \mu(\beta)=2} \mathrm{hol}_\nabla(\partial\beta) \cdot n_\beta \cdot [L]$$

**Definition (Disk potential)**

$$W_L(z_1, z_2, \cdots, z_n) := \sum_{\beta : \mu(\beta)=2} n_\beta \cdot z^{\partial\beta} \qquad W_L : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

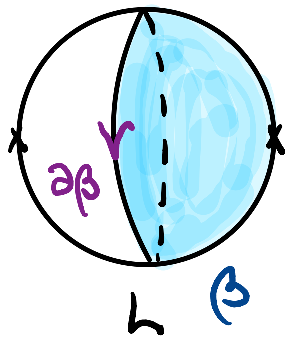
e.g> $(X = \mathbb{C}P^1, \omega_{FS})$, $L = $ (equator)

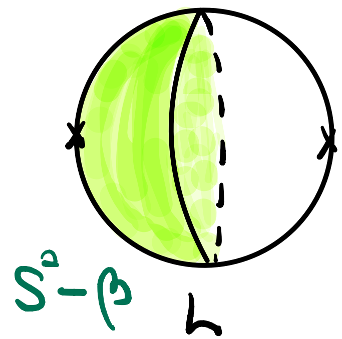
- $\pi_2(X, L) \simeq \mathbb{Z} \cdot \langle \beta, S^2 - \beta \rangle$

  $\Rightarrow \quad n_\beta = 1$ & $n_{S^2-\beta} = 1$.

- $\pi_1(L) \simeq \mathbb{Z} \cdot \langle \partial\beta \rangle \Rightarrow z := \mathrm{hol}_\nabla(\partial\beta) \in \mathbb{C}^*$

- $W(z) = n_\beta \cdot z^{\partial\beta} + n_{S^2-\beta}\, z^{\partial(S^2-\beta)} = z + \dfrac{1}{z}$




Difficulty of computation of disk potentials

- Classification of holom. disks

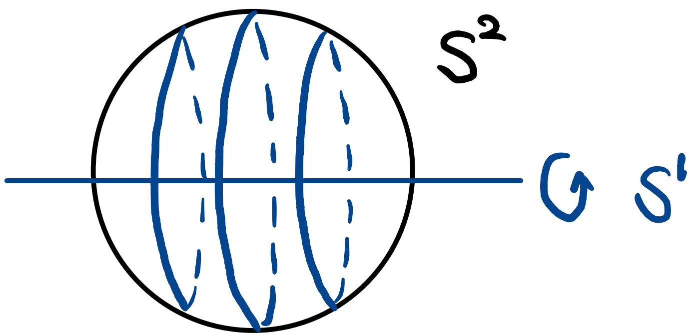
- Computation of open Gromov – Witten invariants

# Disk potentials for toric manifolds/orbifolds

$T^n \curvearrowright (X^{2n}, \omega)$: Symplectic toric mfld

free $T^n$-orbit is a Lagrangian torus.

e.g>




(Cho–Oh) Fano toric case

(Disk potential) = (Givental–Hori–Vafa potential)

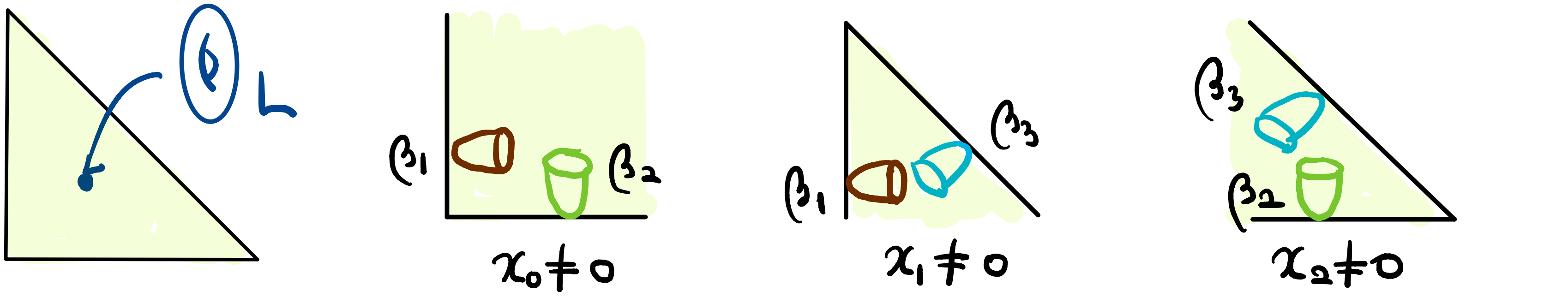
(Fukaya–Oh–Ohta–Ono, Woodward, Cho–Poddar, ⋯)

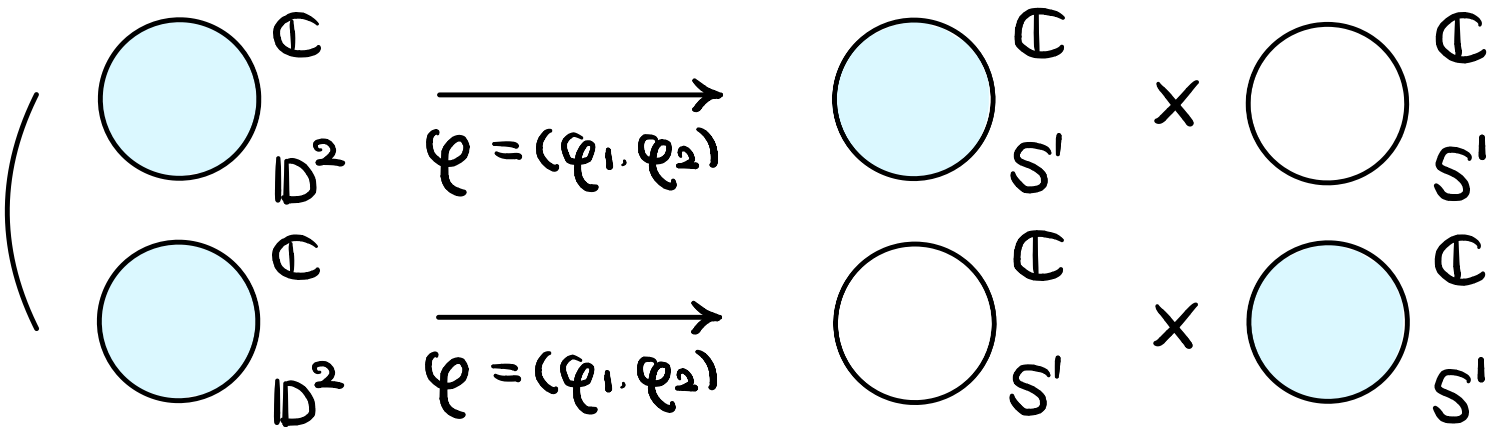
Construct LFT and define disk potentials for arb. cpt toric mflds/orbif.

(Chan–Lau–Leung–Tseng) Semi-Fano toric case e.g.> $(X = \mathbb{CP}^2, \omega_{FS})$, $L = \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 : |x_0| = |x_1| = |x_2|\}$

$T^2 \circlearrowleft (e^{i\theta_1} x_1, e^{i\theta_2} x_2)$

$$\downarrow \quad \Phi([x_0 : x_1 : x_2]) = \left( \frac{|x_1|^2}{|x_0|^2 + |x_1|^2 + |x_2|^2} \quad \frac{|x_2|^2}{|x_0|^2 + |x_1|^2 + |x_2|^2} \right)$$



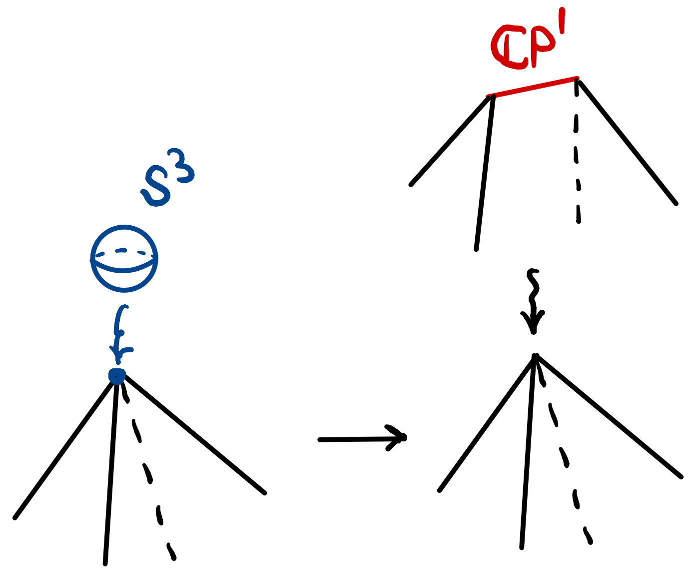

$$W_L(z_1, z_2, z_3) = \sum_{i=1,2,3} n_{\beta_i} z^{\partial \beta_i} = z_1 + z_2 + \frac{1}{z_1 z_2}$$

# Disk potentials and toric degenerations



- Toric degenerations $\mathfrak{X} = \{\mathfrak{X}_t\}_{t\in\mathbb{C}}$ (Nishinou–Nohara–Ueda, Ruan)

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[rr, squiggly, "\Psi_{1,0}"] \arrow[dr, "\Phi_1"'] & & \mathfrak{X}_0 \arrow[dl, "\Phi_0"] \\ & \Delta & \end{tikzcd}$$



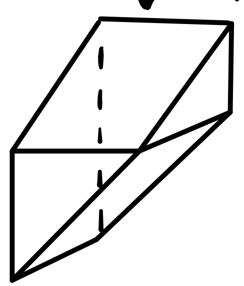

$\circledast$ $\begin{cases} \mathfrak{X}_t : \text{Fano} \\ \mathfrak{X}_0 : \text{(generalized) conifold singularity} \end{cases}$

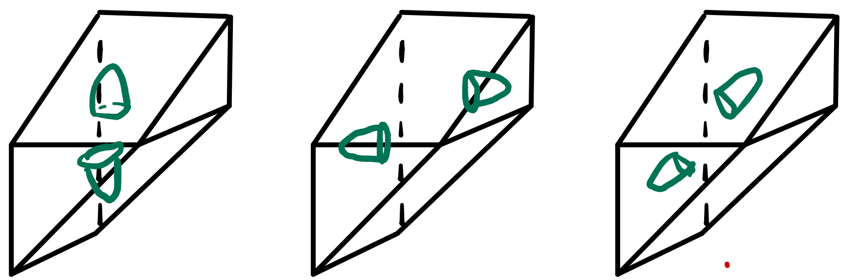
$\Rightarrow W_{disc} = W_{GHV}$ (resemble Fano toric case)

e.g.> Gelfand–Zeitlin systems for partial flag var. $(SL(n,\mathbb{C})/P)$

$X = \mathcal{F}\ell(1,2;3)$

$\downarrow \Phi$





Still far from understanding general toric deg.

Some examples not satisfying ⊛

- Partial flag var. (other than type A/C)

  Gelfand – Zeitlin toric deg. of partial flag var. of type B & D

- Even for partial flag var. of type A.

$$\begin{tikzcd}
 & & \mathcal{X}_1 \arrow[dll, rightsquigarrow] \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] \arrow[drr, rightsquigarrow] & & \\
\mathcal{X}_0 \arrow[d] & \mathcal{X}_0' \arrow[d] & & \mathcal{X}_0'' \arrow[d] & \mathcal{X}_0''' \arrow[d] \\
\Delta & \Delta' & & \Delta'' & \Delta'''
\end{tikzcd}$$

(Cho – K. – Lee – Park)

found some toric deg. not satisfying ⊛.

( String polytopes.
  Newton – Okounkov bodies

- Polygon spaces. $\mathcal{M}_{\vec{r}}$

**Need to enhance understanding beyond ⊛**

# 3. Quadric hypersurfaces

- $Q_n = \left\{ [x_0 : x_1 : \cdots : x_{n+1}] \in \mathbb{C}P^{n+1} : 2x_0x_1 + x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = 0 \right\}$

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n+1} \end{bmatrix} \begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & O_{n\times n} \\ O_{n\times n} & I_{n\times n} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = 2x_0x_1 + x_2^2 + x_3^2 + \cdots + x_{n+1}^2$$

$$\Rightarrow Q_n \simeq OG(1, \mathbb{C}^{n+2}) \simeq G_{\mathbb{C}}/P \simeq G/G\cap P \simeq SO(n+2)/S(O(2)\times O(n))$$

$\left( G_{\mathbb{C}} := SO(n+2;\mathbb{C}),\ G = SO(n+2) \right)$

- $G \curvearrowright \mathfrak{g}^*$ co-adjoint action
  $\lambda \in \mathfrak{g}^*$
  $\rightsquigarrow \mathcal{O}_\lambda^n$ : the co-adjoint orbit

$G \curvearrowright \mathfrak{g} := \{ (n+2)\times(n+2) \text{ skew-symm. matrices} \}$

$$\mathcal{O}_\lambda^n = \left\{ Q\underset{\sim}{D}Q^{-1} : Q \in SO(n+2) \right\} \simeq Q_n$$

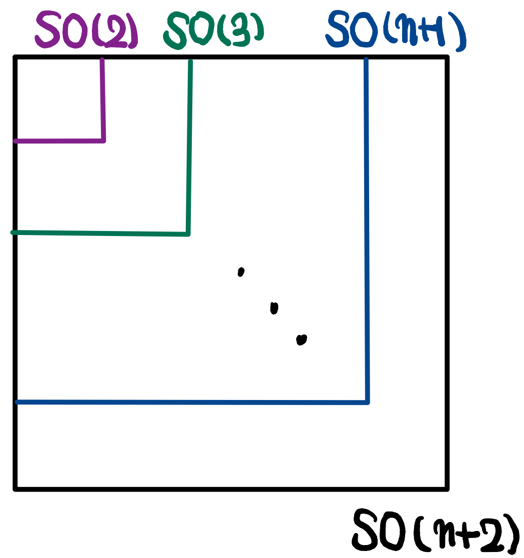
$$D = \begin{bmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & O \\ O & O \end{bmatrix} \quad (\lambda_1 > 0)$$

$$\begin{matrix} \mathfrak{g}^* & \simeq & \mathfrak{g} \\ \circlearrowleft & & \circlearrowleft \\ G & & G \end{matrix}$$

## • Gelfand–Zeitlin systems on $Q_n$

$$SO(n+2) \supseteq SO(n+1) \supseteq SO(n) \supseteq \cdots \supseteq SO(2)$$

$$\begin{array}{ccccccccccc}
\mathcal{O}^n_\lambda & \hookrightarrow & \mathfrak{g}^* := \mathfrak{g}^*_{n+2} & \rightarrow & \mathfrak{g}^*_{n+1} & \rightarrow & \mathfrak{g}^*_n & \rightarrow & \cdots & \rightarrow & \mathfrak{g}^*_2 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & \mathfrak{t}^*_{n+1,+} & & \mathfrak{t}^*_{n,+} & & & & \mathfrak{t}^*_{2,+}
\end{array}$$

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$$\Phi^n_\lambda := (\Phi_1, \Phi_2, \cdots, \Phi_n) : \mathcal{O}^n_\lambda \longrightarrow \mathbb{R}^n$$

$A \in \mathcal{O}^n_\lambda \quad A^{(j)} :=$ the leading $(j \times j)$ principal submatrix of $A$

$$\mathrm{Spec}(A^{(j)}) = \{\pm \lambda^{(j)}_1 \sqrt{-1}, 0, 0, \cdots, 0\} \quad (\lambda^{(j)}_1 \geq 0)$$

$$\Phi_{j-1}(A) := \begin{cases} \lambda^{(j)}_1 & \text{if } j \geq 3 \text{ or } j = 2 \ \& \ \mathrm{Pf}(A^{(2)}) \geq 0 \\ -\lambda^{(2)}_1 & \text{if } j = 2 \ \& \ \mathrm{Pf}(A^{(2)}) < 0 \end{cases}$$

**Theorem** (Thimm & Guillemin–Sternberg)

$\Phi^n_\lambda$ is a completely integrable system (Gelfand–Zeitlin system)

- Perculiar properties of GZ systems

(1) The image is a polytope. (Gelfand – Zeitlin polytope)

(2) Every fiber is a smooth isotropic submfld

(Y. Cho – K. – Oh) describes the GZ fibers in terms of ladder diagrams

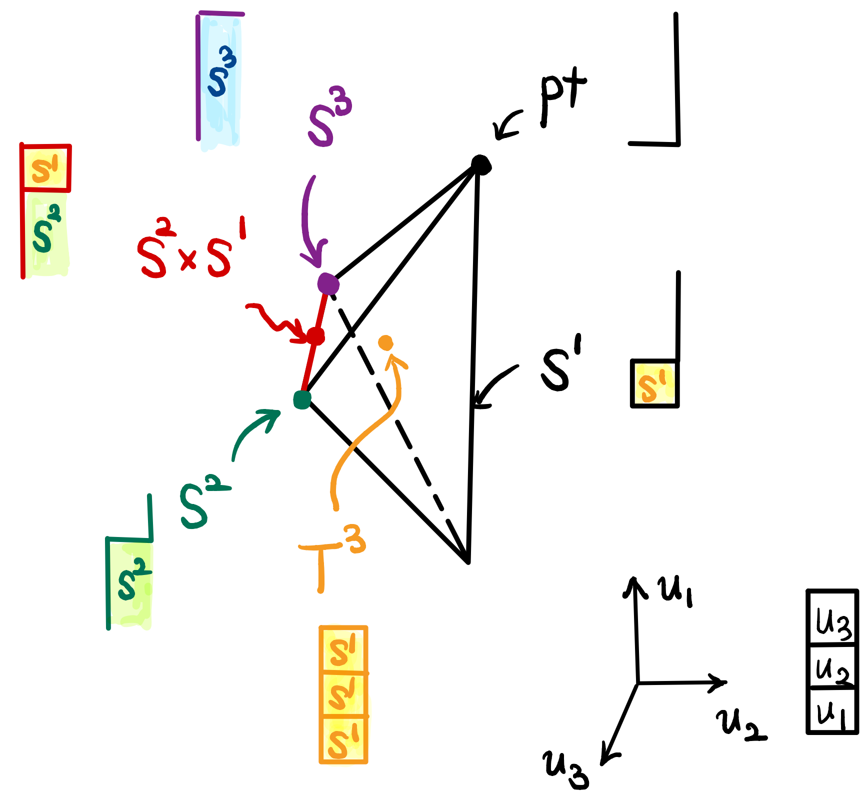
$\mathcal{O}_\lambda^n \simeq OG(1, \mathbb{C}^{n+2}) \simeq Q_n$ e.g.> $Q_3 \simeq OG(1, \mathbb{C}^5)$

- $\Phi_\lambda^n(\mathcal{O}_\lambda^n) =: \Delta_\lambda^n$ is a simplex.

  $\lambda_1 \geq u_n \geq u_{n-1} \geq \cdots \geq u_2 \geq |u_1|$

- The fiber over every pt $\in \mathring{\Delta}_\lambda^n$ is a Lagrangian torus.

- Non-torus Lag. can occur only at the codim two stratum given by $u_2 = 0$



# 4. Disk potentials for quadrics

$\Phi_\lambda^n := (\Phi_1, \Phi_2, \cdots, \Phi_n) : \mathcal{O}_\lambda^n \longrightarrow \mathbb{R}^n$, $L := (\Phi_\lambda^n)^{-1}(u_1 = 0, u_2 = 1, \cdots, u_n = n-1)$

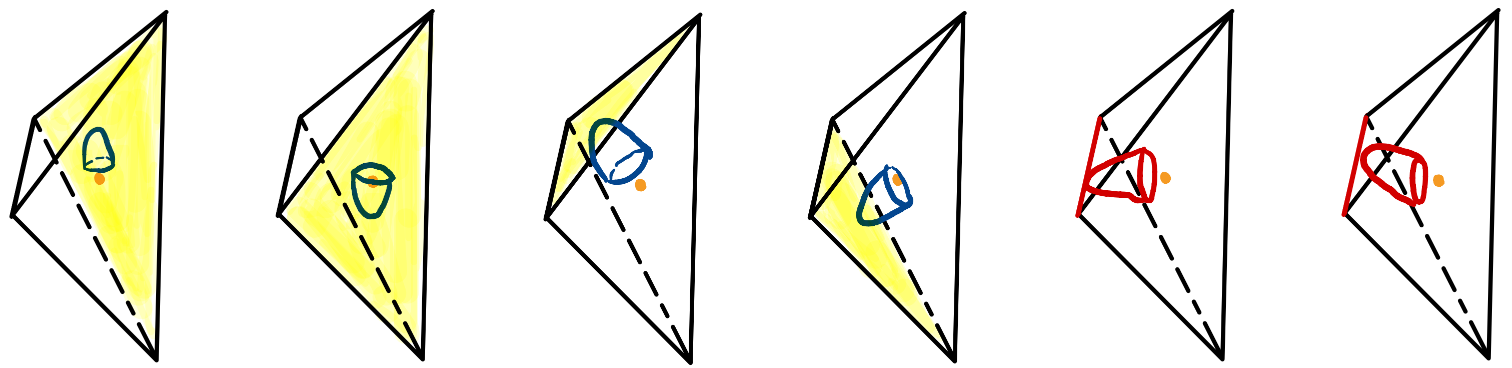
(Guillemin–Sternberg) $\Phi_i$ generates a Ham. $S^1$-action on $\mathring{\mathcal{O}}_\lambda^n := (\Phi_\lambda^n)^{-1}(\mathring{\Delta}_\lambda^n)$

$\theta_i :=$ an (oriented) $S^1$-orbit $\rightsquigarrow \pi_1(L) \simeq \mathbb{Z}\langle \theta_1, \cdots, \theta_n \rangle$. $z_i := \mathrm{hol}_\nabla(\theta_i)$

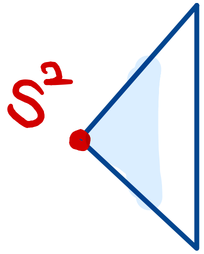
**Theorem (–)**

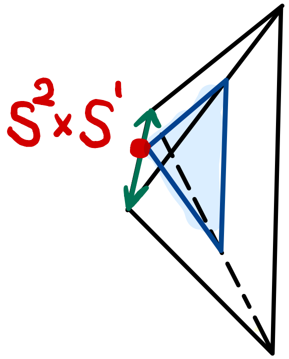
$$W_L(z) = \frac{1}{z_n} + \frac{z_n}{z_{n-1}} + \cdots + \frac{z_3}{z_2} + \frac{z_2}{z_1} + 2z_2 + z_1 z_2$$

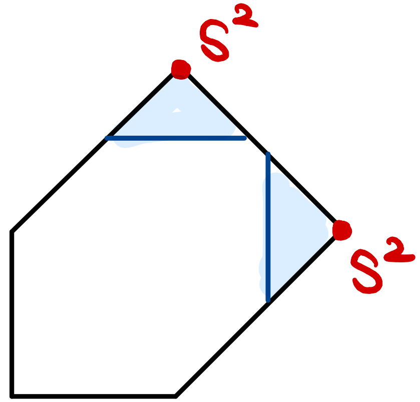
e.g.) $Q_3$ $\quad W_L(z) = \frac{1}{z_3} + \frac{z_3}{z_2} + \frac{z_2}{z_1} + z_1 z_2 + 2z_2$



# Global $\Rightarrow$ Locals $\Rightarrow$ Global$'$



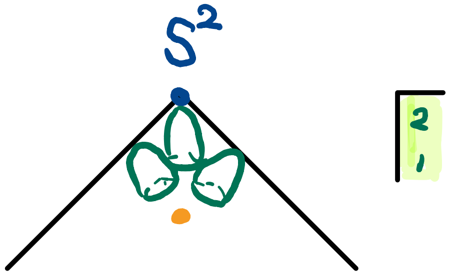

| | | | |
|---|---|---|---|
| <u>Global</u> | $Q_2$ | $OG(1,\mathbb{C}^4)$, $OG(2,\mathbb{C}^5)$ | Partial flag varieties |
| $\Downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| <u>Local</u> | $T^*S^2$ | $T^*S^2$ , $T^*SO(3)$ | $T^*V_k(\mathbb{C}^n)$, $T^*V_k(\mathbb{R}^n)$, $T^*\mathcal{U}(n)$, $T^*\mathcal{O}(n)$ |
| $\Downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| <u>Global$'$</u> | $Q_n$ | Polygon spaces (Lau – Zheng in progress) | ? |

# 5. Further directions (In progress)

- Isotropic flag varieties & Gelfand-Zeitlin systems

Symplectic building blocks includes $T^*SO(n)$, $T^*S^n$, $T^*V_k(\mathbb{R}^n)$, ...

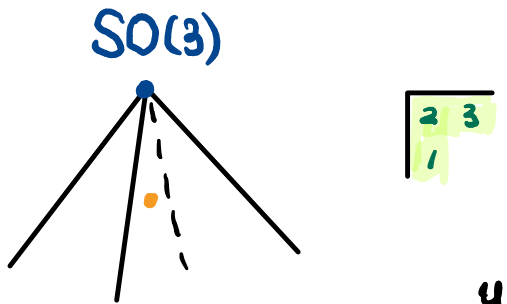
$T^*S^2$




$$W_{loc} = \frac{y_2}{y_1} + 2y_2 + y_1 y_2$$

$$T^*S^2 \hookrightarrow OG(1, \mathbb{C}^4) \simeq Q_2$$

$T^*SO(3)$




$$W_{loc} = y_1 y_2 + y_2 y_3 + \frac{y_2}{y_1} + \frac{y_2}{y_3}$$

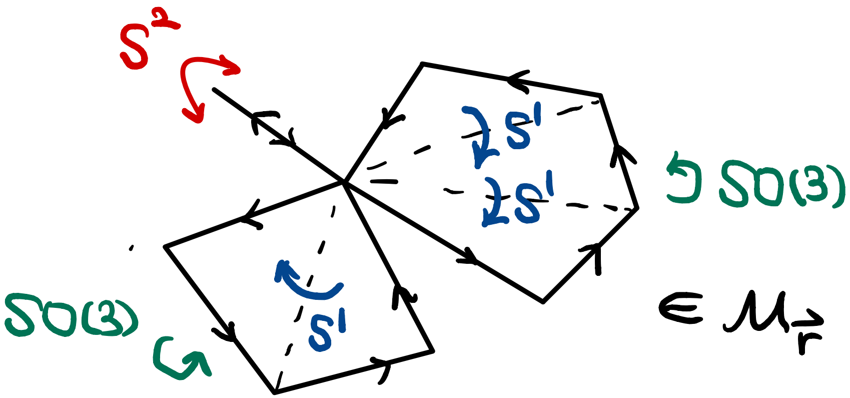
$$T^*SO(3) \hookrightarrow OG(2, \mathbb{C}^5)$$

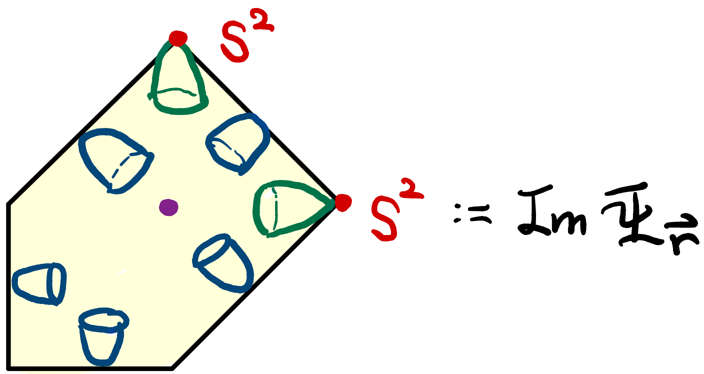
- (w/ Lau–Zheng) Disk potentials for polygon spaces.

  (Hausmann–Knutson) $Gr(2,\mathbb{C}^n) /\!\!/_{\vec{r}}\, T_{U(n)} \simeq \mathcal{M}_{\vec{r}}$ (Polygon space)

  (Kapovich–Millson) $\Psi_{\vec{r}} : \mathcal{M}_{\vec{r}} \longrightarrow \mathbb{R}^{n-3}$. (Bending system)

  (Bouloc) Local models $T^*S^2$, $T^*SO(3)$




Example

e.g.) $n = 2$, $\vec{r} = (1,1,1,1,1)$, $\mathcal{M}_{\vec{r}} \simeq dP_5$.

$$W_{\vec{r}}(\vec{x}) = \left(x_1 + \frac{2}{x_1}\right) + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1 x_2} + \left(x_2 + \frac{2}{x_2}\right)$$

## Sketch of proof (If time permits)

(1) Effective disk classes

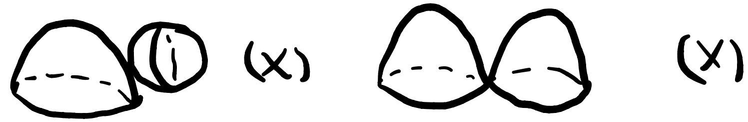
$$\begin{array}{ccc} & & \widehat{\mathcal{X}}_0 \\ & & \vdots \downarrow \\ \mathcal{X}_1 & \rightsquigarrow & \mathcal{X}_0 \end{array}$$

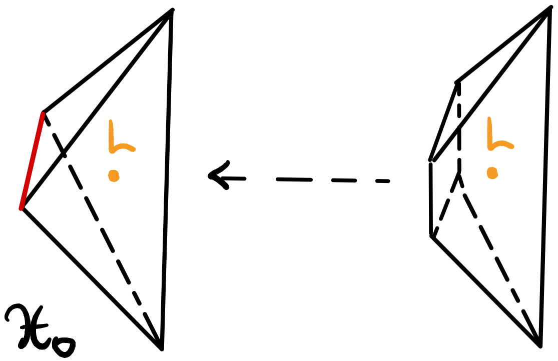
$$\mathcal{X}_t := V(2x_0x_1 + x_2^2 + t(x_3^2 + \cdots + x_{n+1}^2))$$

(Nishinou–Nohara–Ueda, Ruan)

$$\begin{tikzcd} \mathcal{X}_1 \arrow[rr, squiggly, "\varphi"] \arrow[dr, "\Phi_\lambda^n"'] & & \mathcal{X}_0 \arrow[dl, "\Phi_{\text{toric}}"] \\ & \Delta_\lambda^n & \end{tikzcd}$$

Toric deg. of completely int. system

$L$ : monotone & $\mathcal{X}_1$ : minimal Chern # $\geq 2$ $\rightsquigarrow$ 


$\mathcal{X}_0$ $\widehat{\mathcal{X}}_0$

$\beta = [\varphi]$, $\varphi$ intersects (the inv. image of) a face $\geq$ codim 2 (other than / )

$\Rightarrow$ The Maslov index of $\beta \geq 4$

$$\Rightarrow W_L(z) = \frac{1}{z_n} + \frac{z_n}{z_{n-1}} + \cdots + \frac{z_3}{z_2} + \frac{z_2}{z_1} + k z_2 + z_1 z_2$$

(2) Lie theoretical LG mirror of $Q_n$

(Pech–Rietsch–Williams) derive a LG mirror of $Q_n$

| | |
|---|---|
| $QH(Q_n)$ | $\mathrm{Jac}(\check{X}, W)$ |
| $c_1 \cup : QH(Q_n) \longrightarrow QH(Q_n)$ | |
| $\rightsquigarrow \; QH(Q_n) \simeq \bigoplus_{\lambda} QH_\lambda(Q_n)$ | critical values of $W$. |

(3) Structure of monotone Fukaya category

$$W_L(z) = \frac{1}{z_n} + \frac{z_n}{z_{n-1}} + \cdots + \frac{z_3}{z_2} + \frac{z_2}{z_1} + k z_2 + z_1 z_2 \text{ has a critical pt.}$$

$\Rightarrow (L, \nabla)$ is a non-zero object in $\mathrm{Fuk}(Q_n) := \bigoplus_{\lambda} \mathrm{Fuk}_\lambda(Q_n)$.

(Sheridan) $(L, \nabla)$ split-gen. $\mathcal{F}uk_\lambda(X, \omega)$ for some $\lambda$.

We then

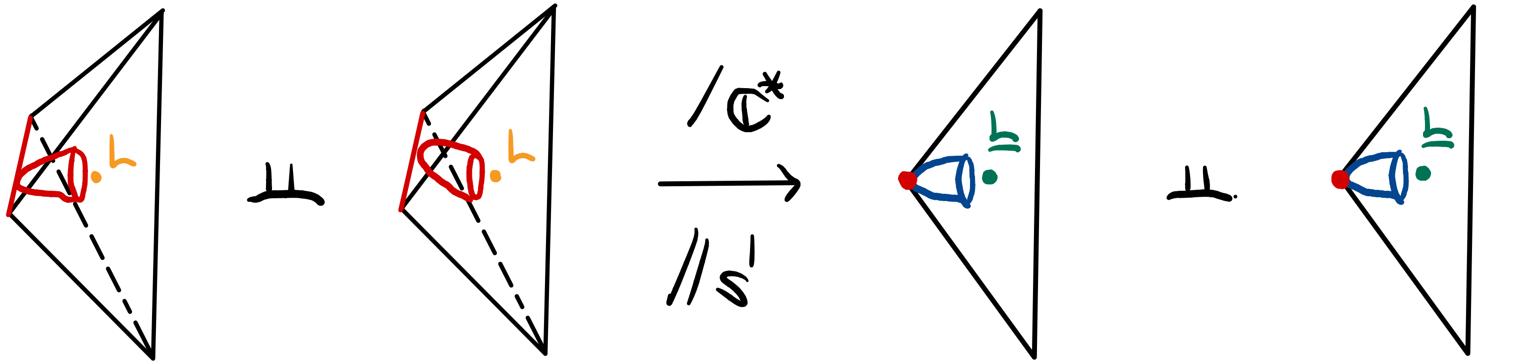
$$\begin{cases} k = 2 & \text{when } n = 2 \\ k = 0 \text{ or } 2 & \text{when } n \geq 3 \end{cases}$$

(4) Disk correspondence btw pre-quotient & quotient.

<u>Lemma</u> $Q_n$ $(n \geq 3)$

$$\mathcal{M}_1(\beta) = \mathcal{M}_1^{(1)}(\beta) \amalg \mathcal{M}_1^{(2)}(\beta) \xrightarrow{\sim} \mathcal{M}_1(\underline{\beta}) \times S^1 \amalg \mathcal{M}_1(\underline{\beta} + S^2) \times S^1$$

$\Rightarrow k = 2$




$\beta$ $\qquad$ $\beta$ $\qquad$ $\underline{\beta}$ $\qquad$ $\underline{\beta} + S^2$

($\Phi_3$ generates a Hamiltonian action)

$\boxed{/}$

<u>Theorem (−)</u>

$$W_L(z) = \frac{1}{z_n} + \frac{z_n}{z_{n-1}} + \cdots + \frac{z_3}{z_2} + \frac{z_2}{z_1} + 2z_2 + z_1 z_2$$

Thank You!