Noncommutative geometry in application to
machine learning

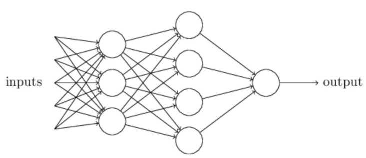
Friday, March 26, 2021 9:05 PM

Neural network in machine learning shares the same starting point as quiver representation theory. In this talk, I will build an algebro-geometric formulation of a `computing machine' which is well-defined over the moduli space. The main algebraic ingredient is extending the associative geometry of Connes, Cuntz-Quillen, Ginzburg to near-rings, which capture the non-linear activation functions in neural network. Furthermore, I will explain a uniformization between spherical, Euclidean and hyperbolic moduli of framed quiver representations.

with George Jeffreys

I. Neural network and quiver representation
II. Uniformization of metrics on framed quiver moduli
III. An AG formulation using noncommutative geometry
IV. Experiments

Neural network and quiver representation



Fix a directed graph Q. Associate to

vertex: vector space
arrow: linear map.

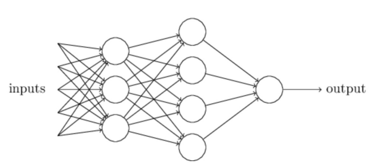
That is, a quiver representation w.

Fix a collection of vertices $i_{\text{in}}, i_{\text{out}}$, and $V_{i_{\text{in}}}, V_{i_{\text{out}}}$.

To approximate any given continuous function $f: K \to V_{i_{\text{out}}}$ (where $K \overset{\text{cpt}}{\subset} V_{i_{\text{in}}}$) by using a representation w.

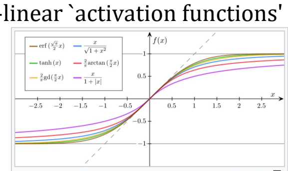
Fix $\gamma \in i_{\text{out}} \cdot \mathbb{C}Q \cdot i_{\text{in}}$.

Get a linear function $f_{\gamma,w}: V_{i_{\text{in}}} \to V_{i_{\text{out}}}$.



Linear approximation $f_{\gamma,w}$ is not good enough!

Introduce non-linear `activation functions' at vertices.



Some sigmoid functions compared. In the drawing all functions are normalized in such a way that their slope at the origin is 1.

Compose with these activation functions and get **network function** $f_{\tilde{\gamma},w}: V_{i_{\text{in}}} \to V_{i_{\text{out}}}$ for every $w \in \text{Rep}(Q)$.

Minimize
$C(V) = |f_{\tilde{\gamma},w} - f|^2_{L^2(K)}$
by taking a (stochastic) gradient descent on the vector space $\mathrm{Rep}(Q)$.

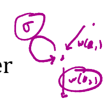
So a neural network is essentially:
a quiver representation, together with
a fixed choice of non-linear functions on the representing vector spaces, and a fixed path.

Relation between quiver representations and neural network was observed by **[Armenta-Jodoin 20]**.

AI neural network has achieved great success in many fields of science and daily life.

Related to lot of areas in math:
Representation theory, stochastic analysis, Riemannian geometry, Morse theory, mathematical physics…

Basic motivating questions:
1. Are there any deeper geometric structures in the subject?
2. Can modern geometry provide new insight for the theory and find enhancement of methods?



Main difference between neural network and quiver representations is:
there are non-linear activation functions.

$\sigma(l_1 + l_2) \neq \sigma(l_1) + \sigma(l_2)$

Definition 1.11. *A near-ring is a set $\tilde{A}$ with two binary operations $+, \circ$ called addition and multiplication such that*

(1) $\tilde{A}$ is a group under addition.
(2) Multiplication is associative.
(3) Right multiplication is distributive over addition:

$$(x + y) \circ z = x \circ z + y \circ z$$

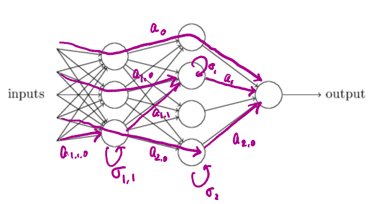
for all $x, y, z \in \tilde{A}$.

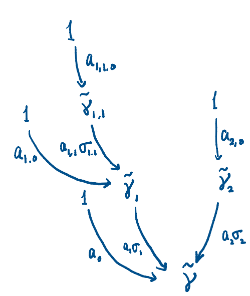
In this paper, the near-ring we use will be required to satisfy that:

(4) $(\tilde{A}, +)$ is a vector space over $\mathbb{F} = \mathbb{C}$, with $c \cdot (x \circ y) = (c \cdot x) \circ y$ for all $c \in \mathbb{C}$ and $x, y \in \tilde{A}$.
(5) There exists $1 \in \tilde{A}$ such that $1 \circ x = x = x \circ 1$.

The quiver path algebra together with symbols for the activation functions forms a **$\mathbb{C}$-near-ring** $A\{\sigma_1, \ldots, \sigma_n\}$.
(Distributive law does not hold on one side.)

$\tilde{\gamma} \in A\{\sigma_1, \ldots, \sigma_n\}$ is encoded by a tree (whose leaves are inserted with 1).
ex. $\tilde{\gamma} = a_0 + a_1\sigma_1 \circ (a_{1,0} + a_{1,1}\sigma_{1,1} \circ a_{1,1,0}) + a_2\sigma_2 \circ a_{2,0}$.
$a_0, a_1, a_{1,0}, a_{1,1}, a_{1,1,0}, a_2, a_{2,0} \in A = \mathbb{C}Q$.





Another gap between quiver and neural network:

In math, we work with **moduli space of representations**:
$\mathcal{M} := \text{Rep}(Q)//_{\chi}\, \text{Aut}$.

Isomorphic objects should produce the same result.

However, **this is not true for $f_{\tilde{\gamma},w}$ given as above**:
Any useful non-linear functions $\sigma: V_i \to V_i$ are NOT equivariant under $\text{GL}(V_i)$:
$\sigma(g \cdot v) \neq g \cdot \sigma(v)$.
Then $f_{\tilde{\gamma},w}$ does not descend to $[w] \in \mathcal{M}$.

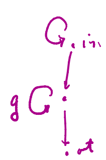
A crucial gap between neural network and representation theory!

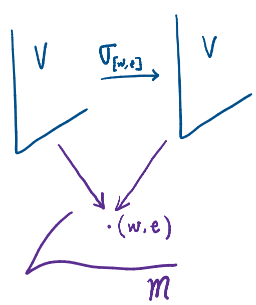
It poses an obstacle for carrying out machine learning using moduli space of quiver representations.

[arXiv:2101.11487] provided a simple solution to overcome this obstacle.

Finding $\sigma: V_i \to V_i$ such that $\sigma(g \cdot v) \neq g \cdot \sigma(v)$ is impossible.

On the other hand, we can find fiber-bundle maps
$\sigma: V_i \times \text{Rep}(Q) \to V_i \times \text{Rep}(Q)$
that satisfies
$g \cdot \sigma_w(v) = \sigma_{g \cdot w}(g \cdot v)$.

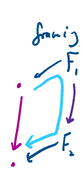




Then $f_{\tilde{\gamma},w}$ will be invariant under group action on the **middle vertices**.

However, still not invariant for GL_d-action at the input and output vertices!

Another key point: use **framing** for quiver representations:



- Inputs and outputs live in the framing vector spaces, which are independent of the internal state spaces.
- By using metrics on the universal bundles, we can use functions on the framing vector spaces to construct fiber-bundle maps on the universal bundles.

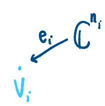
We construct canonical metric on the universal bundle that has explicit algebraic formula.

Rmk.
There is rising interest on relations between geometry and data science.
For instance, [**Lei-Luo-Yau-Gu**] studied manifold structure of data.

Here, we focus on the use of moduli space and metric, and finding an algebraic formulation of a computing machine.

Framed quiver moduli

Fix Q. $A = \mathbb{C}Q$.

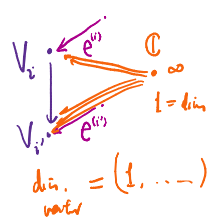


Framed representation:
Vertex: V_i
Arrow: w_a
together with $e_i\colon \mathbb{C}^{n_i} \to V_i$ (called framing).

Framed A-module $\leftrightarrow$ Framed representation:

$$V = \bigoplus_{i \in Q_0} V_i \,.$$

$$\mathrm{Rep}_{\vec{n},\vec{d}} := \mathrm{Rep}_{\vec{d}} \times \bigoplus_{i \in Q_0} \mathrm{Hom}_k(\mathbb{C}^{n_i}, V_i)\,.$$



$\vec{d}$ is dim. of rep.
$\vec{n}$ is dim. of framing.

$$\mathcal{M} := \mathrm{Rep}_{\vec{n},\vec{d}} //_{\chi} \mathrm{GL}_{\vec{d}}$$

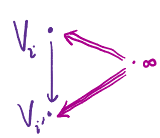
In this case, we have a fine moduli of framed quiver representations which is smooth.
[**Kings; Nakajima; Crawley-Boevey; Reineke**]

Stability condition:
no proper subrepresentation of V contains Im e.

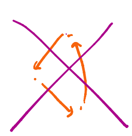
$\mathcal{M}_{\vec{n},\vec{d}} := \{\text{stable framed rep. } (V, e)\}/GL_{\vec{d}}$.

Typical example:
$Gr(n, d)$.

Remark: $\mathcal{M}_{\vec{n},\vec{d}}$ is the usual GIT quotient for a bigger quiver $\hat{Q}$ which has one more vertex ∞ than Q,
together with n_i arrows from ∞ to i.
(Put dim=1 over the vertex ∞. Then take the character $\Theta = -\infty^*$ for slope stability $\Theta(\vec{\alpha})/\Sigma\vec{\alpha}$.)



Topology of $\mathcal{M}_{\vec{n},\vec{d}}$ is well-known.



Thm. [Reineke]
Suppose Q has no oriented cycle. Then $\mathcal{M}_{\vec{n},\vec{d}}$ is an iterated Grassmannian bundle, which can also be identified as a quiver Grassmannian.

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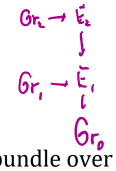
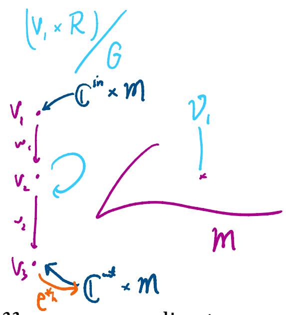




$\mathcal{V}_i$: universal bundle over vertex i.

To run machine learning over $\mathcal{M}_{\vec{n},\vec{d}}$:

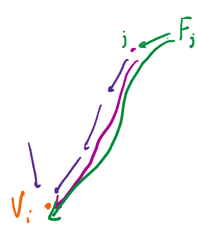
1. Fix a $\mathbb{C}$-near-ring $\mathbb{C}Q\{\sigma_1, \ldots, \sigma_N\}$.
2. Fix $\tilde{\gamma} \in \mathbb{C}Q\{\sigma_1, \ldots, \sigma_N\}$ (an algorithm).
3. $\mathbb{C}Q$ acts on the universal bundles $\mathcal{V}_i$.
4. Fix equivariant fiber-bundle maps $\mathcal{V}_{i(l)} \to \mathcal{V}_{j(l)}$ corresponding to σ_l.
5. At the input vertices i of $\tilde{\gamma}$, compose with framing map e_i.
   At the output vertices j, compose with the adjoint e_j^{*h}.
6. This cooks up a function $f^{\tilde{\gamma}}$ on the framing vector spaces, well-defined over $\mathcal{M}_{\vec{n},\vec{d}}$. Then follow a gradient descent of
   $|f^{\tilde{\gamma}} - f|^2_{L_2(K \subset F)}$
   in $\mathcal{M}_{\vec{n},\vec{d}}$.

For the adjoint, we need Hermitian metric h on the universal bundles $\mathcal{V}_i$.
Moreover, we also need Kaehler metric on the moduli space $\mathcal{M}_{\vec{n},\vec{d}}$.

Rmk.
Formulating as gradient descent on moduli space, this is now a familiar scenario of minimizing energy functional in math. physics.
[**Donaldson; Uhlenbeck-Yau**]
Finding Hermitian Yang-Mills metric on holomorphic vector bundles.

Canonical metric exists for $\mathcal{M}_{\vec{n},\vec{d}}$,
which has an **algebraic expression in terms of the quiver**:

Thm.
For every quiver Q and every $i \in Q_0$,

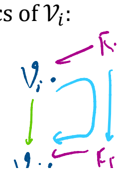


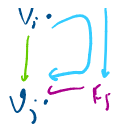
$$H_i: \mathrm{Rep}_{\vec{n},\vec{d}} \to \mathrm{End}(V_i),$$
$$(w,e) \mapsto \left(\sum_{h(\gamma)=i} w_\gamma e_{t(\gamma)} (w_\gamma e_{t(\gamma)})^*\right)^{-1}$$

gives a well-defined metric on $\mathcal{V}_i \to \mathcal{M}$.

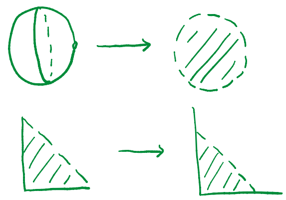
Moreover, if Q has no oriented cycle, the Ricci curvature
$i \sum_i \partial\bar{\partial} \log \det H_i$
of the resulting metric on $\otimes_{i \in Q_0} \mathcal{V}_i$ defines a Kaehler metric on $\mathcal{M}_{\vec{n},\vec{d}}$.

Important observation:
Maps on the framing $F_i \to F_j$ induce
equivariant fiber-bundle maps $\mathcal{V}_i \to \mathcal{V}_j$
using Hermitian metrics of $\mathcal{V}_i$:





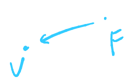
Rmk.
In [arXiv:2101.11487], we show that the symplectomorphism
$$\frac{\vec{z}}{\sqrt{1+|\vec{z}|^2}} : (\mathbb{C}^n, \omega_{\mathbb{P}^n}) \to (B^n, \omega_{\mathrm{std}})$$
can be used as an activation function, in the sense that universal approximation theorem holds.



Summing up, now we have:

$$\tilde{A} \to \mathbb{D}(\mathcal{C}, \mathrm{Map}(F))$$
$$\tilde{\gamma} \mapsto f^{\tilde{\gamma}}_{(w,e)}(v) = He(e_{\mathrm{out}}, \tilde{\gamma} \circ_{[w,e]} e_{\mathrm{in}} \cdot v)$$

Question:
How to relate this moduli formulation
back to the original setup over
Euclidean space of representations?



From now on, let's take $\vec{n} \geq \vec{d}$.

Write the framing as $e^{(i)} = (\epsilon^{(i)} \; b^{(i)})$

By using the quiver automorphism, $\epsilon^{(i)}$ can be made as Id.
whenever $\epsilon^{(i)}$ is invertible.

This gives a chart:
$\mathrm{Rep}_{\vec{n}-\vec{d},\vec{d}} \hookrightarrow \mathcal{M}_{\vec{n},\vec{d}}$.

Restricting the above $He(e_{\mathrm{out}}, \tilde{\gamma} \circ_{[w,e]} e_{\mathrm{in}} \cdot v)$ to this chart,
pretending the metrics are all trivial,
it recovers the usual Euclidean setup!

Does $\mathrm{Rep}_{\vec{n}-\vec{d},\vec{d}} \subset \mathcal{M}_{\vec{n},\vec{d}}$ have a more intrinsic interpretation?

Yes, by considering uniformization.

Uniformization

For $\mathrm{Gr}(n,d) = U(n)/U(d)U(n-d)$,
has Hermitian symmetric dual
$\mathrm{Gr}^{-}(n,d) = U(d,n-d)/U(d)U(n-d)$
$= \{\text{Spacelike subspace in } \mathbb{R}^{d,n-d}\} \overset{\textbf{Borel}}{\subset} \mathrm{Gr}(n,d)$.

Ex. Hyperbolic disc $D \subset \mathbb{CP}^1$.
Hyperbolic <--> spherical.

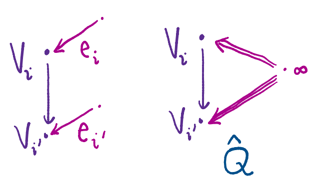
Such symmetric dual and embedding was studied uniformly for general symmetric spaces by [**Chen-Huang-Leung**].

By [**Reineke**], framed quiver moduli $\mathcal{M}_{n,d}$ is an iterated Grassmannian bundle.

What is its `non-compact dual'?

$\hat{Q}$: the quiver with one more vertex denoted as ∞.

Assume $\vec{n} > \vec{d}$. Write $e^{(i)} = (\epsilon^{(i)}\ b^{(i)})$



For each i, define
$$H_i^{-} = \left(\sum_{h(\gamma)=i} (-1)^{s(\gamma)}\gamma\gamma^*\right)^{-1} = \left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & -I_{N_i-d_i}\end{pmatrix} \rho_i^*\right)^{-1}$$
where γ is a path in $\hat{Q}$ with $t(\gamma) = \infty$;
$s(\gamma) = 1$ for $\gamma = \epsilon_j^{(i)}$, and -1 for all other γ.

$R^{-} := \{(w,e) \in R_{n,d} \colon H_i^{-} \text{ is positive definite for all } i\}$.

Lemma.
$\emptyset \neq R^{-} \subset \{(w,e) \colon \epsilon^{(i)} \text{ is invertible } \forall i\} \subset R^{s}$.

Lemma.
R^{-} is G_d-invariant.

$\mathcal{M}^- := R^-/G_d$.
The moduli of space-like framed representations.

Theorem 1.

- H_i^- defines Hermitian metric on the universal bundle $\mathcal{V}_i \to \mathcal{M}^-$.
- $H_{M^-} := -i\partial\bar{\partial}\log\det H_i^-$ defines a Kaehler metric on M^-.

- There exists a (non-holomorphic) isometry, which respects the real structure:
$$(M^-, H_{M^-}) \cong \left(\prod_i \mathrm{Gr}^-(m_i, d_i), \bigoplus_i H_{\mathrm{Gr}^-(m_i,d_i)}\right)$$
where $m_i = n_i + \sum_{a:h(a)=i} \dim V_{t(a)}$.
- There is a canonical identification of $\mathcal{V}_i \to \mathcal{M}^-$ with $\mathcal{V}_{\mathrm{Gr}^-(m_i,d_i)} \to \prod_i \mathrm{Gr}^-(m_i, d_i)$ covering the isometry.

Remark:
$\mathrm{Gr}^-(m,d) = \left\{b \in \mathrm{Mat}_{d\times(m-d)}: bb^* < I_d\right\}$ has non-positive curvature (invariant under parallel transport).

In the same manner like before, have network function
$f_{(w,e)}^{\tilde{\gamma}}(v) = H(e_{out}, \tilde{\gamma} \circ_{[w,e]} e_{in} \cdot v)$
over (M^-, H_T).

Remark:
Machine learning using hyperbolic geometry has recently attracted a lot of research in learning graphs and word embeddings.
Most has focused on taking hyperbolic metric in the fiber direction.

Homogeneous spaces have also been introduced in the fiber direction [**Cohen; Geiger; Weiler**], to make use of symmetry of input data.

Here, we extract **natural Hermitian-symmetric structure for the base moduli space**, which universally exists for all neural network models.

A parallel Euclidean story:
Take
$H_i^0 = \left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & 0 \end{pmatrix} \rho_i^*\right)^{-1}$.
That is, we assign positive sign to $\epsilon_j^{(i)}$ and 0 (instead of -1) to all other paths of $\hat{Q}$.

$R^0 := \{(w,e) \in R_{n,d}\colon H_i^0 \text{ is positive definite for all } i\}$.

Prop.
$R^0//_\chi G_d = \mathrm{Rep}_{n-d,d}$
a vector space.
Also H_i defines trivial metric on $V_i|_{M^0}$.

That is,
$\mathbf{Rep}_{n-d,d} \subset \mathcal{M}_{n,d}$ is the moduli of framed positive-def. representations with respect to H_i^0.

This recovers the usual Euclidean machine learning.

Conclusion:
$\mathcal{M}, \mathcal{M}^-, \mathcal{M}^0$ (spherical, hyperbolic, Euclidean) are the moduli of framed positive-definite representations with respect to
$H_i = (\rho_i \rho_i^*)^{-1}$,
$H_i^- = \left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & -I \end{pmatrix} \rho_i^*\right)^{-1}$,
$H_i^0 = H_i^0 = \left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & 0 \end{pmatrix} \rho_i^*\right)^{-1}$ respectively.

Can connect them in a family:
$\left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & tI \end{pmatrix} \rho_i^*\right)^{-1}$.

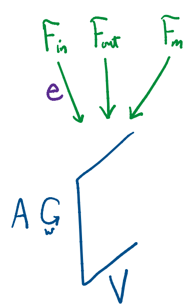
Now, let's go to a more general algebraic viewpoint.

<u>Noncommutative formulation</u>

***A*: associative algebra.**
• consisting of *linear operations* of the machine.

***V*: a vector space (basis-free).**
• States of the machine (before observation).

Consider ***A*-module structures $\boldsymbol{w}\colon \boldsymbol{A} \to \mathbf{gl}(\boldsymbol{V})$**.
• Linear operations on the state space.

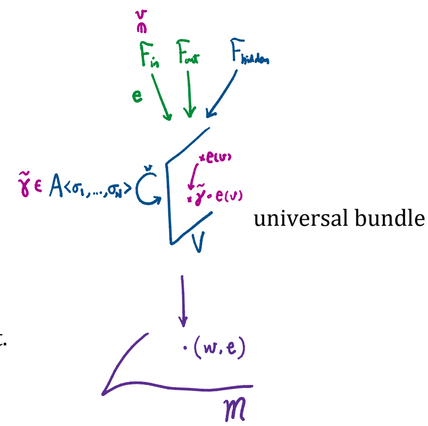


In reality, data are observed and recorded in fixed basis!
Framing *e*:
$F = F_{\text{in}} \oplus F_{\text{out}} \oplus F_m$ (with fixed basis), with linear maps $e\colon F \to V$.
• $F_{\text{in}} \oplus F_{\text{out}}$: vector spaces of all possible inputs and outputs.
• F_m: Physical memory for the machine.
• e: to set up and observe the states.

Get a framed A-module (V, w, e).

Fix $\boldsymbol{\gamma} \in \boldsymbol{A}$.
have $f^\gamma\colon F_{\text{in}} \to F_{\text{out}}$,
$\boldsymbol{f^\gamma(v) := e_{\text{out}}^*(\gamma \cdot e_{\text{in}}(v))}$

- The `machine function'.
  Given an input signal v, send it to machine by e_{in};
  then perform operations according to γ;
  then output by the adjoint of e_{out}.)
- **Metric is needed** to define the adjoint.



Set of framed modules:
$R \coloneqq \{(w,e): w \in A \to \mathfrak{gl}(V) \text{ alg. homo.};\ e \in \text{Hom}(F,V)\}$.

Consider $\mathcal{M} = [R/G]$ where $G = GL(V)$.
Have universal bundle $\mathcal{V}$ descended from $V \times R$.

Equip $\mathcal{V}$ with metric, that is,
a family of metrics $h_{(w,e)}$ on $\boldsymbol{V} \to R$ which is G-equivariant.

Also take
non-linear operations $\sigma_1, \dots, \sigma_N$.

Naively, take the near ring
$A\{\sigma_1, \dots, \sigma_N\}$.

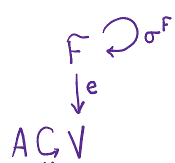
Then an A-module lifts as an $A\{\sigma_1, \dots, \sigma_N\}$-module.

Unfortunately, don't have nice correspondence in the morphism level:
an A-module morphism $\phi: V \to V$ does not respect $A\{\sigma_1, \dots, \sigma_N\}$-module structure:

In particular, do not have
$[R(A)/G] \to R_F(A\{\sigma_1, \dots, \sigma_N\})$.

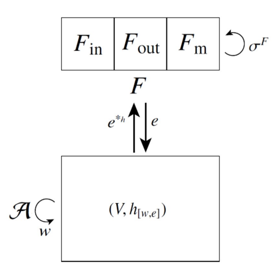
Remedy: make use of framing and metric.

Recall that, σ_i should be treated as non-linear maps on the framing $F \to F$, NOT on V.



Given ANY $\sigma^F: F \to F$, cook up fiber bundle map $\sigma_{(w,e)}$
using the equivariant metric and framing:
$\sigma_{(w,e)}(v) \coloneqq e \cdot \sigma^F\left(h_{(w,e)}(e_1, v), \dots, h_{(w,e)}(e_n, v)\right)$

- Observe and record the state using e, do the non-linear operation, and then send it back as state.



Let's conclude with the following definition.

Def.
An **activation module** consists of:
(1) a noncommutative algebra A and vector spaces V, F;
(2) a collection of possibly non-linear functions
$\sigma_j^F: F \to F$;
(3) A family of metrics $h_{(w,e)}$ on V over the space R of framed A-modules which is $\text{GL}(V)$-equivariant.

Encode non-linear operations by the following **nc near-ring**:

$\tilde{\boldsymbol{A}} \coloneqq \left(\mathbf{Mat}_F\left(\hat{A}^{\text{double}}\right)\right)\{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_N\}$

where

$\hat{A}^{\text{double}}$ is the doubling of $\mathbb{C}\hat{Q}$; (so has e^*, a^*)

$\mathrm{Mat}_F\left(\hat{A}^{\text{double}}\right)$ is an n-by-n matrix, whose entries are cycles in $\mathbb{C}\hat{Q}$ based at the framing vertex ∞.

Doubling is a standard procedure in construction of Nakajima's quiver variety.

Prop. We have

$[R(A)/G] \to R_F(\tilde{A})$

Prop.

Each point in the moduli space $\mathcal{M}$ gives a well-defined map

$\tilde{A} \to \mathrm{Map}(F)$.

That is, we have

$\tilde{A} \to \mathbb{D}(\mathcal{M}, \mathrm{Map}(F))$

Note: $\mathcal{M}$ above is moduli of A-modules, NOT the doubling.

The actions of e^*, a^* on $F \oplus V$ are produced by the adjoint with respect to

h (the equivariant family of metrics on V).

Have **differential forms for nc algebra** $\boldsymbol{A}$

[**Connes; Cuntz-Quillen; Kontsevich**; **Ginzburg...**].

$\mathrm{DR}^*(A) \to \Omega\left(R(A)^G\right)$

Study moduli spaces for all dimension vectors at the same time!

The noncommutative differential forms can be described as follows. Consider the quotient vector space $\overline{A} = A/\mathbb{K}$ (which is no longer an algebra). We think of elements in $\overline{A}$ as differentials. Define

$$D(A) := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} D(A)_n, \ D(A)_n := A \otimes \overline{A} \otimes \ldots \otimes \overline{A}$$

where n copies of $\overline{A}$ appear in $D(A)_n$, and the tensor product is over the ground field $\mathbb{K}$. We should think of elements in $\overline{A}$ as *matrix-valued* differential one-forms. Note that $X \wedge X$ may not be zero, and $X \wedge Y \neq -Y \wedge X$ in general for matrix-valued differential forms X, Y.

The differential $d_n : D(A)_n \to D(A)_{n+1}$ is defined as

$$d_n(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_n}) := 1 \otimes \overline{a_0} \otimes \ldots \otimes \overline{a_n}.$$

The product $D(A)_n \otimes D(A)_{m-1-n} \to D(A)_{m-1}$ is more tricky:

$$(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_n}) \cdot (a_{n+1} \otimes \overline{a_{n+2}} \otimes \ldots \otimes \overline{a_m})$$

$$:= (-1)^n a_0 a_1 \otimes \overline{a_2} \otimes \ldots \otimes \overline{a_m} + \sum_{i=1}^{n} (-1)^{n-i} a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_i a_{i+1}} \otimes \ldots \otimes \overline{a_m} \tag{9}$$

which can be understood by applying the Leibniz rule on the terms $\overline{a_i a_{i+1}}$. Note that we have chosen representatives $a_i \in A$ for $i = 1, \ldots, n+1$ on the RHS, but the sum is independent of choice of representatives (while the product $\overline{a_i a_{i+1}}$ itself depends on representatives).

$d^2 = 0.$

The Karoubi-de Rham complex is defined as

$$DR^\bullet(A) := \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] \tag{10}$$

where $[a, b] := ab - (-1)^{|a||b|} ba$ is the graded commutator for a graded algebra. d descends to be a well-defined differential on $DR^\bullet(A)$. Note that $DR^\bullet(A)$ is not an algebra since $[\Omega^\bullet(A), \Omega^\bullet(A)]$ is not an ideal. $DR^\bullet(A)$ is the non-commutative analog for the space of de Rham forms. Moreover, there is a natural map by taking trace to the space of G-invariant differential forms on the space of representations $R(A)$:

$$DR^\bullet(A) \to \Omega^\bullet(R(A))^G. \tag{11}$$

We extend such notions to the near-ring $\tilde{A}$.

Theorem 1.40. *There exists a degree-preserving map*

$$DR^\bullet(\widetilde{\mathcal{A}}) \to (\Omega^\bullet(R, \mathbf{Map}\,(F,F)))^G$$

which commutes with d on the two sides, and equals to the map (14)*:* $DR^\bullet(\mathrm{Mat}_F(\hat{\mathcal{A}})) \to (\Omega^\bullet(R, \mathrm{End}\,(F)))^G$ *when restricted to* $DR^\bullet(\mathrm{Mat}_F(\hat{\mathcal{A}}))$*. Here,* $\mathbf{Map}\,(F,F)$ *denotes the trivial bundle* $\mathrm{Map}\,(F,F) \times R$*, and the action of* $G = \mathrm{GL}(V)$ *on fiber direction is trivial.*

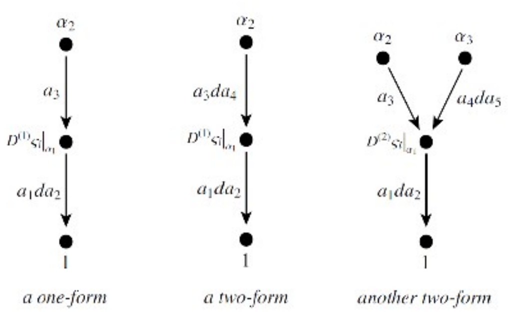


FIGURE 3.

(The number of leaves is required to be ≤ form degree.)
Also have $d^2 = 0$.

In particular, the function

$$\int_K \left| f^{\tilde{\gamma}}_{(w,e)}(v) - f(v) \right|^2 dv$$

and its differential are induced from 0-form and 1-form on $\tilde{A}$.
Central object in machine learning.

Thus the learning is governed by geometric objects on $\tilde{A}$!

Remark:
[**Ginzburg**]: Noncommutative Chern-Weil theory -
replacing Lie algebra $\mathfrak{g}$ by an nc algebra A.

In an ongoing work, we consider
$\tilde{A}$-valued connection and curvatures for fiber bundles.

This has application to recurrent neural network and its higher dimensional analog.

Experiments

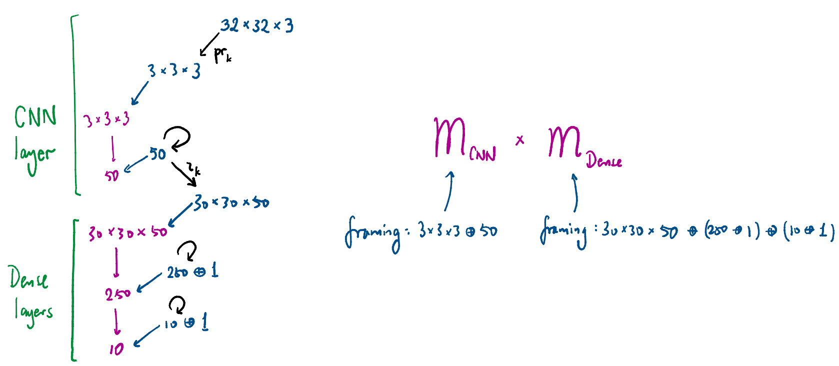
Let's experiment with metrics on the moduli space of representations.

```
from keras.datasets import cifar10
```

```
Using TensorFlow backend.
```

To train machine to classify these pictures into 10 classes.
Want to compare the results of using trivial and non-trivial metrics in the moduli space of framed quiver representations.

ex.



Metric on universal bundles:

$$H_i = (\rho_i \mathcal{I} \rho_i^*)^{-1} = \left(I_{d_i} - \frac{\widetilde{W_i}\widetilde{W_i}^*}{M} \right)^{-1}.$$

Metrics on moduli spaces:

$$h_{\mathcal{M}} = -M \left(\sum_i tr(\rho_i \mathcal{I} \rho_i^*)^{-1} (\partial\rho_i) \mathcal{I} (\partial\rho_i)^* - \sum_i tr(\rho_i \mathcal{I} \rho_i^*)^{-1} \rho_i \mathcal{I} (\partial\rho_i)^* (\rho_i \mathcal{I} \rho_i^*)^{-1} (\partial\rho_i) \mathcal{I} \rho_i^* \right).$$

($M = \infty$ <-> Euclidean; $M > 0$ <-> the non-compact dual $\mathcal{M}^-$; $M < 0$ <-> $\mathcal{M}$.)

Abelianize to simplify the computation:
Take $(\mathbb{C}^\times)^d$ in place of $GL(d)$ in $\mathcal{M} = R/GL(d)$.
This means taking rep. (of a bigger quiver) with dimension vector $(1, \dots, 1)$.
Then metrics on universal bundles are recorded as 1×1 matrices.

The actual model in the experiment:

```
inputs = keras.Input(shape=input_shape)
y = hypConv2D(50, kernel_size=(3, 3),padding='same')(inputs)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = hypConv2D(75, kernel_size=(3, 3),padding='same')(y)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = Dropout(0.25)(y)
y = hypConv2D(125, kernel_size=(3, 3),padding='same')(y)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = Dropout(0.25)(y)
y = layers.Flatten()(y)
y = hypMDenseb(500)(y)
y = Activation(activations.relu)(y)
y = Dropout(0.4)(y)
y = hypMDenseb(250)(y)
y = Activation(activations.relu)(y)
y = Dropout(0.3)(y)
y = hypMDenseb(n_classes)(y)
outputs = layers.Softmax()(y)
model = hypModel(inputs=inputs, outputs=outputs)
model.compile(optimizer="adam", loss="categorical_crossentropy", metrics=["accuracy"])
history = model.fit(x_train, y_train, batch_size=128, epochs=50, validation_split=0.1)
```

```
inputs = keras.Input(shape=input_shape)
y = EuclidConv2D(50, kernel_size=(3, 3),padding='same')(inputs)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = EuclidConv2D(75, kernel_size=(3, 3),padding='same')(y)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = Dropout(0.25)(y)
y = EuclidConv2D(125, kernel_size=(3, 3),padding='same')(y)
y = layers.MaxPooling2D(pool_size=(2, 2))(y)
y = Dropout(0.25)(y)
y = layers.Flatten()(y)
y = Denseb(500)(y)
y = Activation(activations.relu)(y)
y = Dropout(0.4)(y)
y = Denseb(250)(y)
y = Activation(activations.relu)(y)
y = Dropout(0.3)(y)
y = Denseb(num_classes)(y)
outputs = layers.Softmax()(y)
model = EuclidModel(inputs=inputs, outputs=outputs)
model.compile(optimizer="adam", loss="categorical_crossentropy", metrics=["accuracy"])
history = model.fit(x_train, y_train, batch_size=128, epochs=50, validation_split=0.1)
```

```
def call(self, x):
    Hinv = 1 - tf.math.reduce_sum(tf.math.square(self.kernel),[0,1,2]) / self.M
    y = K.conv2d(x, self.kernel,padding=self.padding)
    return keras.activations.relu(y/Hinv)
```

```
#hyperbolic gradient for 1st conv2d layer
#g_i = H_i (Id - H_i wtilde_i wtilde_i^*)
#g_i^(-1) wtilde_i = partial_i /H_i - (partial_i dot wtilde_i) wtilde_i/(M+|wtilde_i|
H1inv = 1 - tf.math.reduce_sum(tf.math.square(trainable_vars[0]),[0,1,2]) / M1
grads[0] = grads[0] * H1inv \
- tf.multiply(tf.reduce_sum(tf.multiply(trainable_vars[0],grads[0]),[0,1,2]),\
              trainable_vars[0]) \
/ (M1+tf.divide(tf.reduce_sum(tf.square(trainable_vars[0]),[0,1,2]),H1inv))
```

$$H_i = (\rho_i \mathcal{J} \rho_i^*)^{-1} = \left(I_{d_i} - \frac{\widetilde{w_i}\widetilde{w_i}^*}{M} \right)^{-1}$$

$$= \left(1 - \frac{|\widetilde{w_i}|^2}{M} \right)^{-1} \text{ if } d_i = 1.$$

$$h_{\mathcal{M}} = -M \cdot \left(\sum_i tr\left((\rho_i \mathcal{J} \rho_i^*)^{-1} (\partial \rho_i) \mathcal{J} (\partial \rho_i)^*\right) + \sum_i tr\left((\rho_i \mathcal{J} \rho_i^*)^{-1} \rho_i \mathcal{J} (\partial \rho_i)^* (\rho\right.\right.$$

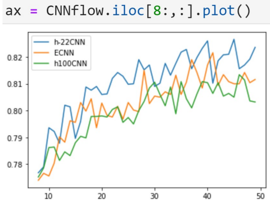
$$h^{\mathcal{M}}_{\widetilde{w}^{0,1}_{kj} \widetilde{w}^{1,0}_{qp}} = \bigoplus_i H^{(i)}_{kj} \left(\delta_{jp} + \frac{1}{M} \cdot \widetilde{w}_p^* \cdot H^{(i)} \cdot \widetilde{w}_j \right).$$

After Abelianize:

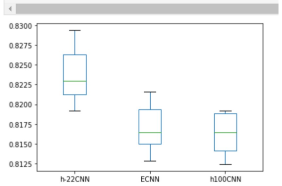
$$h^{\mathcal{M}}_l = H_l \left(I + \frac{1}{M} H_l \widetilde{w}_l^* \widetilde{w}_l \right).$$

$$(\text{grad } f)_l = \frac{1}{H_l} \partial_{\widetilde{w}_l} f - \frac{(\partial_{\widetilde{w}_l} f \cdot \widetilde{w}_l^*) \widetilde{w}_l}{M + |\widetilde{w}_l|^2 H_l}.$$

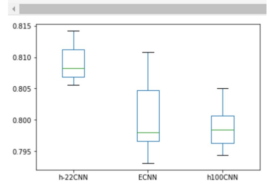
```
ax = CNNflow.iloc[8:,:].plot()
```




```
ax = CNNValAcc.plot.box()
```



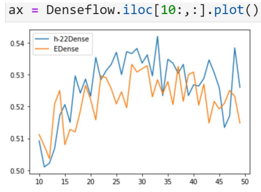

```
ax = CNNtestAcc.plot.box()
```



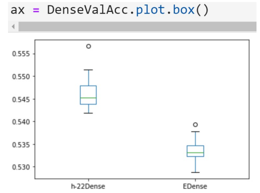
Another test:
Use only dense layers for the same dataset.
Compare trivial and non-trivial metrics.

```
initM = float(-30)
inputs = keras.Input(shape=input_shape)
y = layers.Flatten()(inputs)
y = hypMDenseb(500)(y)
y = Activation(activations.relu)(y)
y = hypMDenseb(250)(y)
y = Activation(activations.relu)(y)
y = hypMDenseb(n_classes)(y)
outputs = layers.Softmax()(y)
model = hypModel(inputs=inputs, outputs=outputs)
model.compile(optimizer="adam", loss="categorical_crossentropy", metrics=["accuracy"])
history = model.fit(x_train, y_train, batch_size=128, epochs=50, validation_split=0.1)
```

```
ax = Denseflow.iloc[10:,:].plot()
```




```
ax = DenseValAcc.plot.box()
```



Conclusion:
in this case, $M < 0$ (curvature ≥ 0) behaves around 1% better than $M = 0$ and $M > 0$.

- The method of moduli spaces and their non-compact duals is UNIVERSAL and works in practice

- Geometric structures on near-ring $\tilde{A}$ is a new subject and govern machine learning over the moduli

- To lay the algebraic foundation of computing machine, and find new applications of geometry.

```
c:\>Thank you for listening_
```