Interleaving distance for sheaves and symplectic geometry

(Joint work with T. Asano)

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Outline of the talk

Microlocal theory of sheaves
(Morse theory for sheaves)

Microsupport

Tamarkin category $\mathcal{D}(M)$
Non-displaceability [Tamarkin '08]

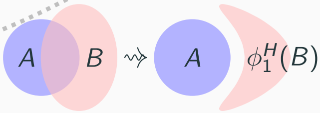
[Kashiwara–Schapira '18]

Our work

Quantitative version of Tamarkin's theorem

Symplectic geometry

Displacement energy



Topological Data Analysis

Persistence modules

C^0-symplectic geometry

Displacement energy in symplectic geometry

M manifold, T^*M cotangent bundle w/ local coord. $(x;\xi)$, $\omega = \sum_i d\xi_i \wedge dx_i$

I open interval $\supset [0,1]$, $H = (H_s)_{s\in I} \colon T^*M \times I \xrightarrow{C^\infty} \mathbb{R}$ w/ compact support

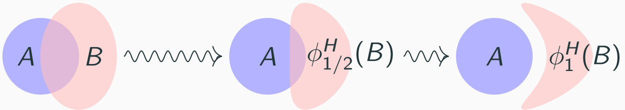
$\phi^H = (\phi^H_s \colon T^*M \xrightarrow{\sim} T^*M)_{s\in I}$ flow generated by H **Hamiltonian isotopy**

Known: 0_M the zero-section of T^*M, M compact $\Rightarrow 0_M \cap \phi^H_1(0_M) \neq \emptyset$

Question: Given $A, B \subset T^*M$ compact subsets, $\exists ? H$ s.t. $A \cap \phi^H_1(B) = \emptyset$ or how "large" does H need to be if $A \cap \phi^H_1(B) = \emptyset$?

$$\|H\|_{\mathrm{osc}} := \int_0^1 (\max_p H_s(p) - \min_p H_s(p))ds$$

$e(A,B) := \inf\{\|H\|_{\mathrm{osc}} \mid A \cap \phi^H_1(B) = \emptyset\}$ **displacement energy**



Persistence homology and persistence modules

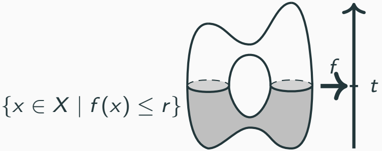
$\mathbf{k}$ field, X topological space, $f\colon X \to \mathbb{R}$ function

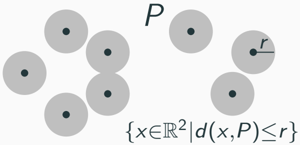
The family of sublevel homology $H_i(\{x \in X \mid f(x) \leq r\})$ satisfies

$$\begin{tikzcd} H_i(\{f \leq r\};\mathbf{k}) \arrow[r] \arrow[rr, bend right] \arrow[r, phantom, shift right=4ex, "\circlearrowright" near end] & H_i(\{f \leq s\};\mathbf{k}) \arrow[r] & H_i(\{f \leq t\};\mathbf{k}) \end{tikzcd} \qquad (r \leq s \leq t)$$

persistent homology

e.g. $X = \mathbb{R}^d \supset P$ finite subset, $f(x) = d(x, P) \rightsquigarrow \{f \leq r\} = \bigcup_{p \in P} \overline{B(p; r)}$





Persistence module: algebraic structure of persistent homology

$r \in \mathbb{R} \rightsquigarrow$ $\mathbf{k}$-vector sp. V_r, $r \leq s \rightsquigarrow$ linear map $V_{r,s}\colon V_r \to V_s$ s.t. $V_{r,r} = \mathrm{id}_{V_r}$

and $$\begin{tikzcd} V_r \arrow[r] \arrow[rr, bend right] \arrow[r, phantom, shift right=4ex, "\circlearrowright" near end] & V_s \arrow[r] & V_t \end{tikzcd} \quad (r \leq s \leq t)$$

$$\mathbb{V} = (V_r, V_{r,s})_{r \leq s}$$

Interleaving distance and stability

Introduce a distance for PMs to describe the robustness for input perturbation

$$\|f-g\|_\infty \le \varepsilon \Rightarrow \begin{tikzcd} H_i(\{f \le r\}) \arrow[r] \arrow[dr] & H_i(\{g \le r+\varepsilon\}) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & H_i(\{f \le r+2\varepsilon\}) \end{tikzcd} \quad \begin{tikzcd} H_i(\{g \le r\}) \arrow[r] \arrow[dr] & H_i(\{f \le r+\varepsilon\}) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & H_i(\{g \le r+2\varepsilon\}) \end{tikzcd}$$

Definition [Chazal et al. '09]

$\varepsilon \in \mathbb{R}_{\ge 0}$, $\mathbb{V} = (V_r, V_{r,s})_{r \le s}$ and $\mathbb{W} = (W_r, W_{r,s})_{r \le s}$ are ε-**interleaved**

$:\Leftrightarrow \exists(\phi_r \colon V_r \to W_{r+\varepsilon})_r, \exists(\psi_r \colon W_r \to V_{r+\varepsilon})_r$ compatible w/ $V_{r,s}$ and $W_{r,s}$ s.t.

$$\begin{tikzcd} V_r \arrow[r, "\phi_r"] \arrow[dr, "V_{r,r+2\varepsilon}"'] & W_{r+\varepsilon} \arrow[d, "\psi_{r+\varepsilon}"] \arrow[dl, phantom, "\circlearrowright"] \\ & V_{r+2\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} W_r \arrow[r, "\psi_r"] \arrow[dr, "W_{r,r+2\varepsilon}"'] & V_{r+\varepsilon} \arrow[d, "\phi_{r+\varepsilon}"] \arrow[dl, phantom, "\circlearrowright"] \\ & W_{r+2\varepsilon} \end{tikzcd}$$

$d_I(\mathbb{V}, \mathbb{W}) := \inf\{\varepsilon \in \mathbb{R}_{\ge 0} \mid \mathbb{V} \text{ and } \mathbb{W} \text{ are } \varepsilon\text{-interleaved}\}$ **interleaving distance**

$\rightsquigarrow d_I(H_i(f), H_i(g)) \le \|f-g\|_\infty$ stability for C^0-distance

Sheaf theory meets persistence modules

Curry '14 and **Kashiwara–Schapira '18** interpreted PMs with sheaf theory

$\mathbf{k}_X$ the constant sheaf on X with stalk $\mathbf{k}$

$\mathrm{D}(\mathbf{k}_X)$ the derived category of sheaves of $\mathbf{k}$-vector spaces

Roughly, **PMs parameterized by $\mathbb{R}$** $\leftrightarrow$ **sheaves on $\mathbb{R}$**

To condier only the positive move, one can use microsupports of sheaves.

$$\begin{tikzcd} V_r \arrow[r] \arrow[rr, bend right] & V_s \arrow[r] \arrow[d, phantom, "\circlearrowright"] & V_t \\ & {} & \end{tikzcd} \quad (r \leq s \leq t)$$

Microsupport: $\mathrm{D}(\mathbf{k}_X) \ni F \mapsto \mathrm{SS}(F) \subset T^*X$ describes the singularity of F

$i_! \mathbf{k}_{[0,1]} \longmapsto$ 

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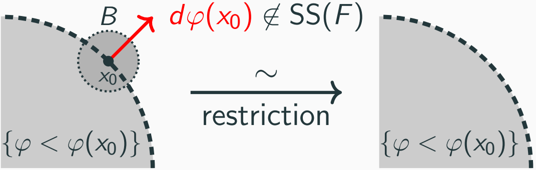
Microsupports of sheaves

Microlocal $=$ local on cotangent bundles

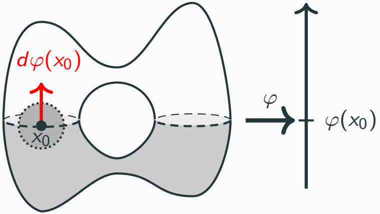
Definition (microsupport)

X manifold, $F \in \mathrm{D}(\mathbf{k}_X)$, $\mathrm{SS}(F) \subset T^*X$ closed $\mathbb{R}_{>0}$-**conic** subset is defined by

$$\mathrm{SS}(F) := \overline{\left\{ (x_0;\xi_0) \in T^*X \;\middle|\; \begin{array}{l} \exists \varphi \colon X \xrightarrow{C^\infty} \mathbb{R} \text{ s.t. } d\varphi(x_0) = \xi_0 \\ \text{and } R\Gamma_{\{x \in X \mid \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} \not\simeq 0 \end{array} \right\}}$$



$R\Gamma(\{\varphi < \varphi(x_0)\} \cup B; F)$ $\qquad$ $R\Gamma(\{\varphi < \varphi(x_0)\}; F)$

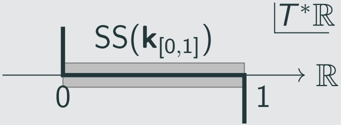


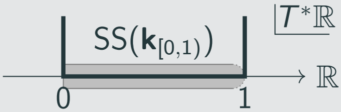
Examples of microsupports of sheaves

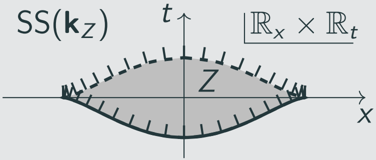
$\mathrm{SS}(F)$ describes the singular codirections of F

Notation: $Z \overset{i}{\hookrightarrow} X$ locally closed subset, $\mathbf{k}_Z := i_! \mathbf{k}_Z$ on X

Examples

1. $\mathrm{SS}(F) \subset 0_X \Leftrightarrow H^j(F)$ locally constant for all $j \in \mathbb{Z}$
2. $M \subset X$ closed submanifold $\Rightarrow \mathrm{SS}(\mathbf{k}_M) = T^*_M X$ conormal bundle
3. 

4. 

5. 

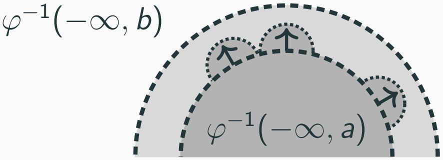
Properties of microsupports

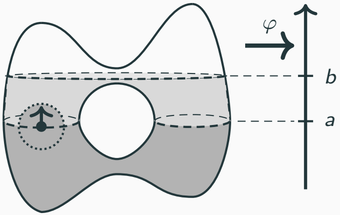
Microlocal theory of sheaves Morse theory for sheaves

Microlocal Morse lemma

$F \in \mathrm{D}(\mathbf{k}_X)$, $\varphi\colon X \xrightarrow{C^\infty} \mathbb{R}$ proper on $\mathrm{Supp}(F)$, $a < b \in \mathbb{R}$
$d\varphi(x) \notin \mathrm{SS}(F)$ for any $x \in \varphi^{-1}[a,b)$

$$\Rightarrow \quad \mathrm{R}\Gamma(\varphi^{-1}(-\infty,b);F) \xrightarrow{\sim} \mathrm{R}\Gamma(\varphi^{-1}(-\infty,a);F)$$





Remark: $\exists$ bounds for SS of $\mathrm{R}f_!$, f^{-1}, $\mathrm{R}\mathcal{H}om$, $\otimes$, etc.

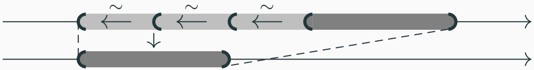
Interleaving for sheaves

$(t;\tau)$ homog. coord. on $T^*\mathbb{R}_t$, $\mathrm{D}_{\tau\geq 0}(\mathbf{k}_{\mathbb{R}_t}) := \{F \in \mathrm{D}(\mathbf{k}_{\mathbb{R}_t}) \mid \mathrm{SS}(F) \subset \{\tau \geq 0\}\}$,
$T_c \colon \mathbb{R}_t \to \mathbb{R}_t,\, t \mapsto t + c$ shift map

$F \in \mathrm{D}_{\tau\geq 0}(\mathbf{k}_{\mathbb{R}_t}),\, c \in \mathbb{R}_{\geq 0} \rightsquigarrow \exists \tau_{0,c}(F) \colon F \to T_{c*}F$ canonical morphism

$R\Gamma((r,s);F)$

$R\Gamma((r,s);T_{c*}F) = R\Gamma((r-c,s-c);F)$



Definition (special case of [Kashiwara–Schapira '18])

$F, G \in \mathrm{D}_{\tau\geq 0}(\mathbf{k}_{\mathbb{R}_t}),\, \varepsilon \in \mathbb{R}_{\geq 0}$

F and G are ε-**isomorphic** $:\Leftrightarrow \exists \alpha \colon F \to T_{\varepsilon *}G, \exists \beta \colon G \to T_{\varepsilon *}F$ s.t.

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\tau_{0,2\varepsilon}(F)"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & T_{\varepsilon *}G \arrow[d, "T_{\varepsilon *}\beta"] \\ & T_{2\varepsilon *}F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\beta"] \arrow[dr, "\tau_{0,2\varepsilon}(G)"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & T_{\varepsilon *}F \arrow[d, "T_{\varepsilon *}\alpha"] \\ & T_{2\varepsilon *}G \end{tikzcd}$$

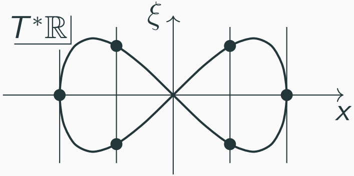
$$d_C(F,G) := \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid F \text{ and } G \text{ are } \varepsilon\text{-isomorphic}\}$$

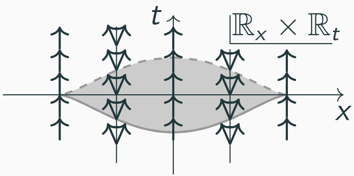
Tamarkin’s cone trick

Rough idea of [Tamarkin ’08] (cf. [Nadler–Zaslow ’09]):
Properties of $A \subset T^*M$ can be deduced from $F \in \mathrm{D}(\mathbf{k}_M)$ w/ $\mathrm{SS}(F) \subset A$
Issue: Cannot treat non-conic subsets with SS
Solution: Add one variable $\mathbb{R}_t$ and conify subsets

$(t;\tau)$ homogeneous coordinates on $T^*\mathbb{R}_t$
$A \subset T^*M$ closed subset
$\rightsquigarrow \mathrm{cone}(A) := \{(x,t;\xi,\tau) \mid \tau > 0, (x;\xi/\tau) \in A\} \subset T^*(M \times \mathbb{R}_t)$ **conic!**

Example





Tamarkin category and separation theorem: $A \cap B \neq \emptyset$?

Definition (Tamarkin category [Tamarkin '08])

$$\mathcal{D}(M) := {}^{\perp}\{F \in \mathrm{D}(\mathbf{k}_{M \times \mathbb{R}_t}) \mid \mathrm{SS}(F) \subset \{\tau \leq 0\}\} \subset \mathrm{D}_{\tau \geq 0}(\mathbf{k}_{M \times \mathbb{R}_t})$$

$A \subset T^*M$ closed subset

$$\mathcal{D}_A(M) := \{F \in \mathcal{D}(M) \mid \mathrm{SS}(F) \subset \overline{\mathrm{cone}(A)}\}$$

Tamarkin's separation theorem

$A, B \subset T^*M$ compact subsets

$$A \cap B = \emptyset \Rightarrow \mathrm{Hom}_{\mathcal{D}(M)}(F, G) \simeq 0 \text{ for any } F \in \mathcal{D}_A(M), G \in \mathcal{D}_B(M)$$

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Tamarkin's separation theorem

$A, B \subset T^*M$ compact subsets

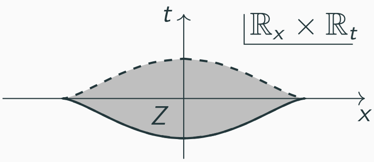
$$A \cap B = \emptyset \Rightarrow \mathrm{Hom}_{\mathcal{D}(M)}(F, G) \simeq 0 \text{ for any } F \in \mathcal{D}_A(M), G \in \mathcal{D}_B(M)$$

Idea: $\exists\, \mathcal{H}om^{\star}(F, G) \in \mathcal{D}(M)$ s.t. $\mathrm{Hom}(F, G) \simeq H^0_{[0,+\infty)}(\mathbb{R}_t; Rq_* \mathcal{H}om^{\star}(F, G))$, where $q \colon M \times \mathbb{R}_t \to \mathbb{R}_t$, $A \cap B = \emptyset \Rightarrow \mathrm{SS}(Rq_* \mathcal{H}om^{\star}(F, G)) \subset 0_{\mathbb{R}}$

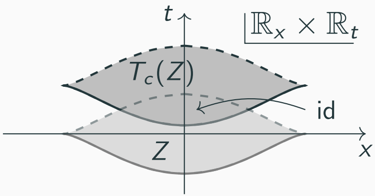
Our result: $e(A,B) \geq ?$

$T_c\colon M \times \mathbb{R}_t \to M \times \mathbb{R}_t, (x,t) \mapsto (x, t+c)$ shift map

$F \in \mathcal{D}(M) \subset \mathrm{D}_{\tau \geq 0}(\mathbf{k}_{M \times \mathbb{R}_t}), c \in \mathbb{R}_{\geq 0} \rightsquigarrow \exists \tau_{0,c}(F)\colon F \to T_{c*}F$



$\mathbf{k}_Z \in \mathcal{D}(\mathbb{R}_x)$



$\tau_{0,c}(\mathbf{k}_Z)\colon \mathbf{k}_Z \to T_{c*}\mathbf{k}_Z$

Our result [Asano.–I. '20]

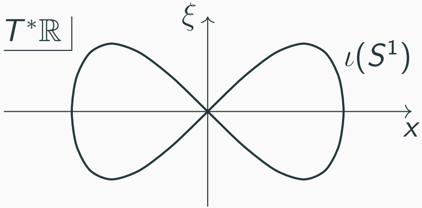
For $F \in \mathcal{D}_A(M), G \in \mathcal{D}_B(M)$,

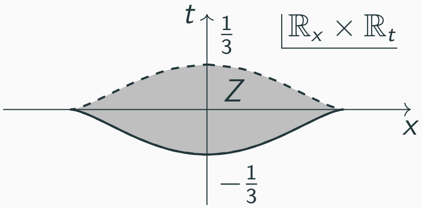
$$e(A,B) \geq \inf\{c \in \mathbb{R}_{\geq 0} \mid \mathrm{Hom}_{\mathcal{D}(M)}(F,G) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \text{ is zero}\}.$$

0-case = **separation theorem**, $+\infty$-case = **non-displaceability theorem**

Example

$\iota\colon S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \to \mathbb{R}^2 \simeq T^*\mathbb{R}, (x,y) \mapsto (x; yx)$





We have seen that $F := \mathbf{k}_Z \in \mathcal{D}_{\iota(S^1)}(\mathbb{R})$. Hence, our result shows

$$\begin{aligned} e(\iota(S^1), \iota(S^1)) &\geq \inf\{c \in \mathbb{R}_{\geq 0} \mid \mathrm{Hom}_{\mathcal{D}(\mathbb{R})}(F, F) \to \mathrm{Hom}_{\mathcal{D}(\mathbb{R})}(F, T_{c*}F) \text{ is zero}\} \\ &= 2/3 \text{ (the same estimate as [Akaho '15])}. \end{aligned}$$

We constructed good sheaves for rational Lagrangian immersions [Asano–I. '20].

Idea of the proof

For a Hamiltonian function $H\colon T^*M \times I \xrightarrow{C^\infty} \mathbb{R}$, $\exists \Phi_1^H\colon \mathcal{D}(M) \to \mathcal{D}(M)$ functor, which restricts to $\mathcal{D}_B(M) \to \mathcal{D}_{\phi_1^H(B)}(M)$ [Guillermou–Kashiwara–Schapira '12]

$$\begin{tikzcd} \mathcal{D}_B(M) \arrow[r, phantom, "\subset"] & \mathcal{D}(M) \circlearrowleft \Phi_1^H \\ B \arrow[u, squiggly] \arrow[r, phantom, "\subset"] & T^*M \circlearrowleft \phi_1^H \arrow[u, squiggly] \end{tikzcd}$$

$\mathcal{D}_B(M) \subset \mathcal{D}(M) \circlearrowleft \Phi_1^H$	**sheaf category**	$G \mapsto \Phi_1^H(G)$ **cost=?**
$B \subset T^*M \circlearrowleft \phi_1^H$	**cotangent bundle**	$B \mapsto \phi_1^H(B)$ cost=$\|H\|_{\mathrm{osc}}$

Idea: Use an **interleaving-like distance** with respect to $\mathbb{R}_t$-variable on $\mathcal{D}(M)$ to measure the difference between G and $\Phi_1^H(G)$

Interleaving-like distance and stability result

Definition [Asano–I. '20] (cf. [Kashiwara–Schapira '18])

$F, G \in \mathcal{D}(M)$, $a, b \in \mathbb{R}_{\geq 0}$

(F, G) (a, b)-**isomorphic** $:\Leftrightarrow \exists \alpha \colon F \to T_{a*}G, \exists \beta \colon G \to T_{b*}F$ s.t.

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\tau_{0,a+b}(F)"'] & T_{a*}G \arrow[d, "T_{a*}\beta"] \arrow[dl, phantom, "\circlearrowleft" near start] \\ & T_{a+b*}F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\beta"] \arrow[dr, "\tau_{0,a+b}(G)"'] & T_{b*}F \arrow[d, "T_{b*}\alpha"] \arrow[dl, phantom, "\circlearrowleft" near start] \\ & T_{a+b*}G \end{tikzcd}$$

$$d_{\mathcal{D}(M)}(F, G) := \inf\{a + b \mid a, b \in \mathbb{R}_{\geq 0}, (F, G) \text{ is } (a, b)\text{-isomorphic}\}$$

Theorem [Asano.–I. '20] (weak form)

For any $G \in \mathcal{D}(M)$, $d_{\mathcal{D}(M)}(G, \Phi_1^H(G)) \leq 2\|H\|_{\mathrm{osc}}$.

Sheaf-theoretic energy estimate

Corollary (energy estimate, weaker form)

$A, B \subset T^*M$ compact subsets, $F \in \mathcal{D}_A(M)$, $G \in \mathcal{D}_B(M)$,

$$2e(A, B) \geq \inf\{c \in \mathbb{R}_{\geq 0} \mid \mathrm{Hom}_{\mathcal{D}(M)}(F, G) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \text{ is zero}\}$$

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Theorem $\Rightarrow$ Corollary: Assume $A \cap \phi_1^H(B) = \emptyset$. For $c > 2\|H\|_{\mathrm{osc}}$ there exist

$$\alpha\colon G \to T_{a*}\Phi_1^H(G) \text{ and } \beta\colon \Phi_1^H(G) \to T_{b*}G$$

satisfying the condition above with $c = a + b$ (Theorem). Then

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(M)}(F,G) \arrow[r, "\alpha"'] \arrow[rr, bend left, "\tau_{0,c}(G)"] & \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{a*}\Phi_1^H(G)) \arrow[r, "T_{a*}\beta"'] \arrow[d, equal, "\text{(separation)}"] & \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G). \\
& 0 &
\end{tikzcd}$$

Completeness of the distance and taking limits

Taking limits for Lagrangians would be interesting and important.
(C^0-symplectic geometry, Gromov-Hausdorff limit of Fukaya categories, ...)

Theorem [Asano.–I. '22] and [Guillermou–Viterbo '22]

$(\mathcal{D}(M), d_{\mathcal{D}(M)})$ is complete.

Corollary

$(H_n)_{n\in\mathbb{N}}$ sequence of Hamiltonian functions, $H_n \xrightarrow{C^0} H_\infty \in C^0(T^*M \times I)$. Then,

$$\exists \Phi_1^{H_\infty} : \mathcal{D}(M) \to \mathcal{D}(M).$$

Moreover, if $\phi_1^{H_n} \xrightarrow{C^0} \varphi_\infty \in \mathrm{Homeo}(T^*M)$, then $\Phi_1^{H_\infty}$ restricts to

$$\Phi_1^{H_\infty} : \mathcal{D}_A(M) \to \mathcal{D}_{\varphi_\infty(A)}(M).$$

φ_∞ as above is called a **Hamiltonian homeomorphism**.

Application: C^0-Arnold-type theorem via sheaves

Recall: 0_M the zero-section of T^*M, $\varphi\colon T^*M \to T^*M$ Hamiltonian diffeo
$\Rightarrow \#(0_M \cap \varphi(0_M)) \geq \mathrm{cl}(M)+1$, if $0_M \pitchfork \varphi(0_M)$, $\#(0_M \cap \varphi(0_M)) \geq \sum_j b_j(M)$

φ_∞ Hamiltonian homeo generated by H_∞
$\Rightarrow$ one can define the set of spectral invariants $\mathrm{Spec}(\varphi_\infty, 0_M) \subset \mathbb{R}$ by continuity

Theorem [Buhovsky–Humilière–Seyfaddini '19] and [Asano–I. '22]

If $\#\,\mathrm{Spec}(\varphi_\infty, 0_M) \leq \mathrm{cl}(M)$, then $0_M \cap \varphi_\infty(0_M)$ is cohomologically non-trivial, in particular, it is infinite.

Strategy: $\mathbf{k}_{M\times[0,\infty)}$ corresponds to 0_M (sheaf quantization of 0_M)

$\Rightarrow$ $\Phi_1^{H_\infty}(\mathbf{k}_{M\times[0,\infty)}) \in \mathcal{D}_{\varphi_\infty(0_M)}(M)$ (sheaf quantization of $\varphi_\infty(0_M)$)

Apply Lusternik–Schnirelmann theory for sheaves to $\Phi_1^{H_\infty}(\mathbf{k}_{M\times[0,\infty)})$.

Summary

1. Introduce an **interleaving-like distance** $d_{\mathcal{D}(M)}$ on Tamarkin category
2. **Stability result** w.r.t. Hamiltonian deformation of sheaves
$$d_{\mathcal{D}(M)}(G, \Phi_1^H(G)) \leq 2\|H\|_{\mathrm{osc}} \quad (G \in \mathcal{D}(M)).$$
3. Corollary: **sheaf-theoretic bound for displacement energy**
4. **Completeness** of the distance $d_{\mathcal{D}(M)}$
5. Application: **Arnold-type theorem for Hamiltonian homeomorphism**

TDA $\hookrightarrow$ **Sheaf theory**

Symplectic geometry $\hookrightarrow$ **Sheaf theory**

Lagrangian L $\dashrightarrow$ sheaf quantization F_L

Fuk(W) over Novikov ring $\dashrightarrow$ $\mathcal{D}(M)$ equipped with $d_{\mathcal{D}(M)}$ (or $\mathcal{D}(M)$ over Novikov ring)

GH limit of Fuk $\dashrightarrow$ limit w.r.t. $d_{\mathcal{D}(M)}$

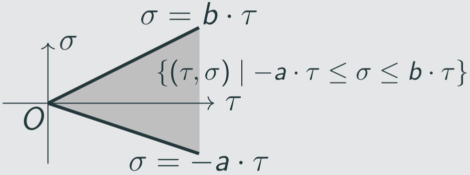
in progress w/ T. Kuwagaki

Outline of the proof of the stability result

Key lemma (weak form)

$\mathcal{H} \in \mathrm{D}(\mathbf{k}_{M\times\mathbb{R}_t\times I})$, $s < s' \in I$.

$$\mathrm{SS}(\mathcal{H}|_{M\times\mathbb{R}_t\times[s,s']}) \subset T^*M \times (\mathbb{R}_t \times I) \times$$



$\Rightarrow d_{\mathcal{D}(M)}(\mathcal{H}|_{M\times\mathbb{R}_t\times\{s\}}, \mathcal{H}|_{M\times\mathbb{R}_t\times\{s'\}}) \le \mathbf{2}(a+b)(s'-s)$.

[GKS '12] $\rightsquigarrow \exists \mathcal{H} \in \mathrm{Sh}(M \times \mathbb{R}_t \times I)$ s.t. $\mathcal{H}|_{M\times\mathbb{R}_t\times\{0\}} \simeq G, \mathcal{H}|_{M\times\mathbb{R}_t\times\{1\}} \simeq \Phi_1^H(G)$,

$$\mathrm{SS}(\mathcal{H}) \subset T^*M \times \left\{ (t,s;\tau,\sigma) \;\middle|\; -\max_p H_s(p)\cdot\tau \le \sigma \le -\min_p H_s(p)\cdot\tau \right\}.$$

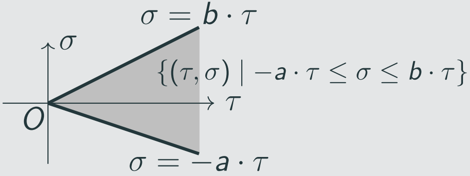
Using Key lemma and Riemann sum, we obtain the theorem.

Outline of the proof of the stability result

Key lemma (weak form)

$\mathcal{H} \in \mathrm{D}(\mathbf{k}_{M\times\mathbb{R}_t\times I})$, $s < s' \in I$.

$$\mathrm{SS}(\mathcal{H}|_{M\times\mathbb{R}_t\times[s,s']}) \subset T^*M\times(\mathbb{R}_t\times I)\times$$



$$\Rightarrow d_{\mathcal{D}(M)}(\mathcal{H}|_{M\times\mathbb{R}_t\times\{s\}}, \mathcal{H}|_{M\times\mathbb{R}_t\times\{s'\}}) \le 2(a+b)(s'-s).$$

$$d_{\mathcal{D}(M)}(\mathcal{H}|_{M\times\mathbb{R}_t\times\{0\}}, \mathcal{H}|_{M\times\mathbb{R}_t\times\{1\}}) \le 2\sum_{k=0}^{n-1}\frac{1}{n}\cdot\max_{s\in[\frac{k}{n},\frac{k+1}{n}]}\left(\max_p H_s(p) - \min_p H_s(p)\right)$$

$$\to 2\int_0^1\left(\max_p H_s(p) - \min_p H_s(p)\right) ds\ (n\to\infty)$$