BU – Keio – Tsinghua Workshop 2022

Exotic monotone Lagrangian tori in flag varieties

(Based on work in progress with
Yunhyung Cho, Myungho Kim, Euiyong Park)

Yoosik Kim
(Pusan National Univ.)

Monotone Lagrangian Submanifolds

(X, ω): symplectic manifold

L: Lagrangian submanifold

(Oh) introduced monotone Lagrangian submanifolds

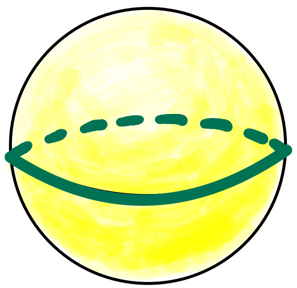
developed Lagrangian Floer theory for monotone Lagrangian submanifolds.

- Two group homomorphisms

$$\begin{cases} \omega: \pi_2(X, L) \to \mathbb{R} & \text{(symplectic area)} \\ \mu: \pi_2(X, L) \to \mathbb{Z} & \text{(Maslov index)} \end{cases}$$

<u>Definition</u> L is called monotone if $\exists\, c > 0$ s.t. $\omega(\beta) = c \cdot \mu(\beta)$ for all $\beta \in \pi_2(X, L)$

e.g>



monotone

non-monotone

Classification Problems

Question Classify all Lagrangian submanifolds of (X, ω)

Almost impossible to attack (b/c there are too many Lagrangian submflds)

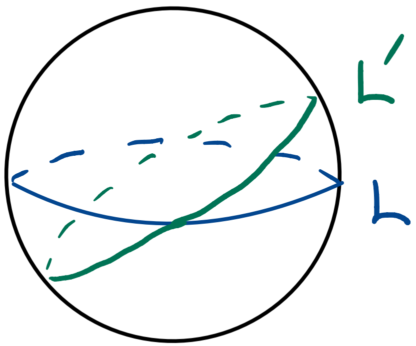
Better Question Classify all meaningful submanifolds of (X, ω) in LFT

- Classify all *monotone* Lagrangian tori up to Hamiltonian isotopy
- Construct *monotone* Lagrangian tori that are not related by Hamiltonian isotopy

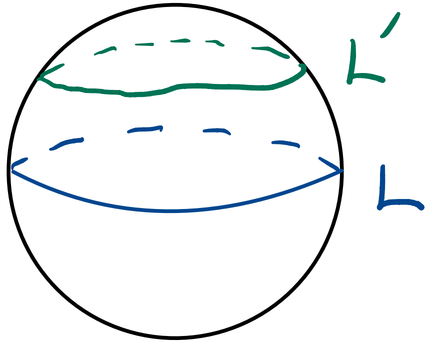
$H : [0,1] \times X \to \mathbb{R}$

$\to dH(\cdot) = \omega(X_H, \cdot)$

$\to \phi_t$ is a flow of X_H (a Hamiltonian isotopy)



Hamiltonian isotopy (o)

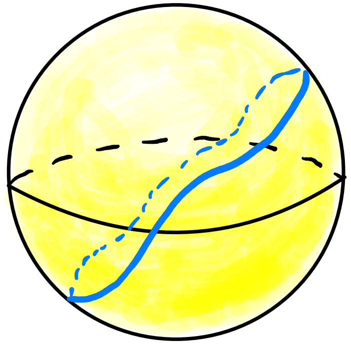


Hamiltonian isotopy (X)

Classification Problems

e.g> $(\mathbb{C}P^1, \omega_{FS})$

Every smooth simple closed curve bisecting the area is Hamiltonian isotopic to a great circle



e.g> $(\mathbb{C}P^2, \omega_{FS})$. $T_{Cliff} = \{[z] \in \mathbb{C}P^2 : |z_0| = |z_1| = |z_2|\}$

(Chekanov) constructed a monotone Lag. torus which is not Hamiltonian isotopic to T_{Cliff}

(Vianna) constructed ∞ many monotone Lag. tori which are not Hamiltonian isotopic to each other.

$\mathbb{C}P^1 \simeq \mathcal{F}\ell(1;2)$

In this talk, we generalize this story to flag var.

Question

Are there infinitely many distinct monotone Lag. tori in $\mathcal{F}\ell(1,2,\cdots,n-1;n)$?

Main Result

Theorem (CKKP)

Every monotone flag manifold $\mathcal{F}\ell(1, 2, \cdots, n-1; n)$ $(n \geq 6)$ admits infinitely many monotone Lagrangian tori that are not Hamiltonian isotopic to each other.

Construction of Lagrangian Torus Fibrations on Flag Manifold

No global Hamiltonian torus symmetry on flag mflds

$$T^N \not\curvearrowright \mathcal{F}\ell(1,2,\dots,n-1;n) \text{ effective} \quad (N := \dim_{\mathbb{C}} \mathcal{F}\ell(1,2,\dots,n-1;n))$$

Should make a choice ($\rightsquigarrow$ make a story even more interesting)

- Choose data for determining a Newton–Okounkov body

$\boxed{\text{Newton–Okounkov body}}$

$\Downarrow$

$\boxed{\text{Toric degenerations}}$

$\Downarrow$

$\boxed{\text{Completely integrable systems}}$

$\Downarrow$

$\boxed{\text{Lagrangian torus}}$

String Polytopes

- Representation theory

  (Berenstein–Zelevinsky, Littelmann) introduced string polytopes

- Newton–Okounkov bodies

  (Kaveh) showed that every string polytope is a Newton–Okounkov body of flag varieties.

- Toric degenerations

  (Caldero, Anderson) $\Rightarrow$ toric deg. of flag var. associated to each string polytope Δ

  $\left(\{\mathcal{X}_t\}_{t\in\mathbb{C}} : \text{ a family of algebraic varieties.} \quad \begin{array}{l} \mathcal{X}_t \simeq \mathcal{F}\ell(1,2,\cdots,n-1;n) \\ \mathcal{X}_0 \simeq \text{Toric variety ass. to } \Delta \end{array} \right)$

- Completely integrable systems

  (Ruan, Nishinou–Nohara–Ueda, Harada–Kaveh)

  $\Rightarrow$ completely integrable system $\Phi : \mathcal{F}\ell(1,2,\cdots,n-1;n) \to \Delta$ s.t.

$$\begin{tikzcd} \mathcal{X}_1 \arrow[rr] \arrow[dr, "\Phi"'] & \arrow[d, phantom, "\circlearrowright"] & \mathcal{X}_0 \arrow[dl, "\Phi_{Toric}"] \\ & \Delta & \end{tikzcd}$$

$\boxed{\text{A Lagrangian torus is located at each point of the interior of } \Delta}$

Choices for String Polytopes

Want to produce a combinatorial gadget to study flag manifolds.

$G = SL(n, \mathbb{C})$.

$W \simeq S_n$ the Weyl group of G

Fact. Every permutation can be expressed as a product of simple transpositions

e.g.> $\sigma = (1\ 3) = (\underline{1}\ 2)(\underline{2}\ 3)(\underline{1}\ 2) = (1\ 2\ 1)$

Choose

(1) $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_{n-1} \varpi_{n-1}$ dominant weight

(Take $\lambda_1 = \cdots = \lambda_{n-1} = 2 \Rightarrow$ *monotone* flag variety)

(2) $\mathbf{i}$ = a reduced expression of the longest element.

A reduced expression of σ is an exp. of σ in terms of simple transp. with *min*. length

S_n has a unique longest element

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} = (1\ 2\ 1\ 3\ 2\ 1\ 4\ 3\ 2\ 1 \cdots n{-}1\ n{-}2 \cdots 2\ 1)$$

$\boxed{\text{The different choice of } \mathbf{i} \text{ produces different Newton – Okounkov bodies for the flag}}$

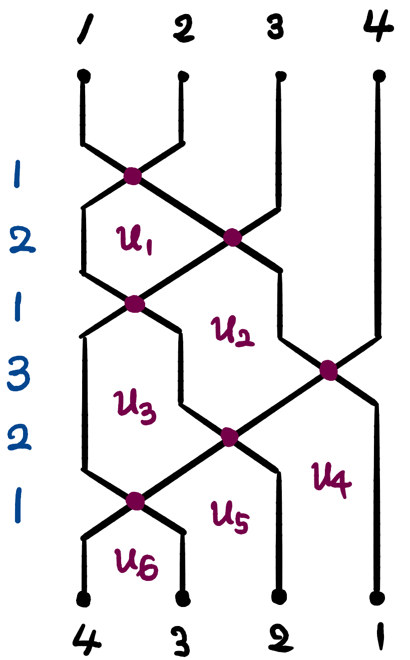
Wiring diagrams

$\mathbf{i}$: a reduced expression of the longest element of S_n

$\mathbf{i} \rightsquigarrow G(\mathbf{i})$ Wiring diagram

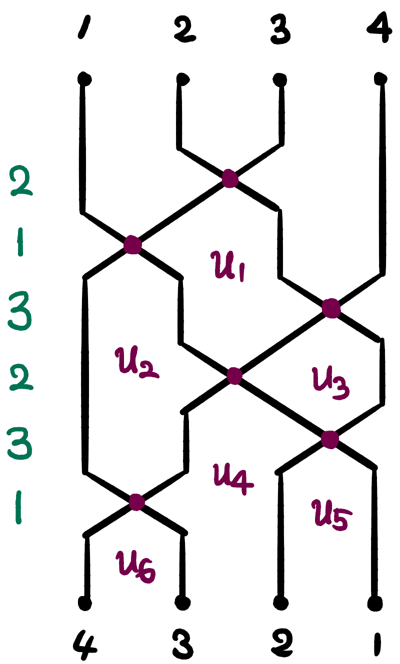
e.g> $n=4$

$\mathbf{i} = (1.2.1.3.2.1)$



$G(\mathbf{i})$

$\mathbf{i}' = (2.1.3.2.3.1)$



$G(\mathbf{i}')$

Oriented wiring diagrams

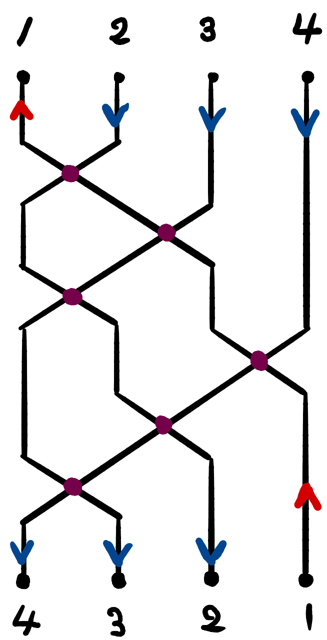
$G(\mathbf{i})$ Wiring diagram

$\Downarrow$

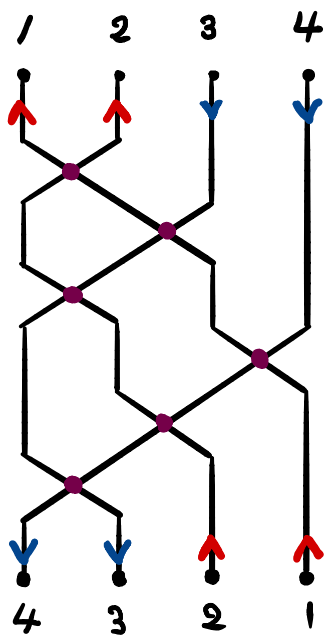
$G(\mathbf{i}, k)$ $(k = 1, 2, \cdots, n-1)$ oriented

e.g> $n=4$

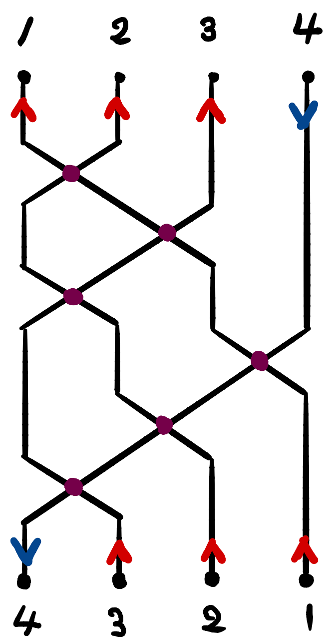
$\mathbf{i} = (1.2.1.3.2.1)$



$G(\mathbf{i}, 1)$



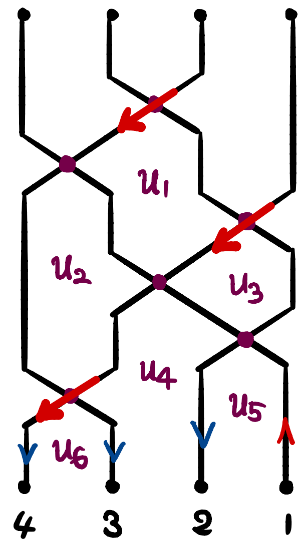
$G(\mathbf{i}, 2)$



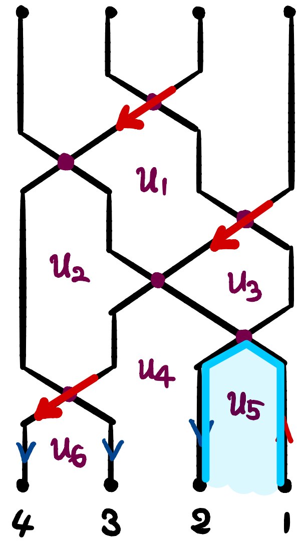
$G(\mathbf{i}, 3)$

Gleizer – Postnikov's description for String Cones

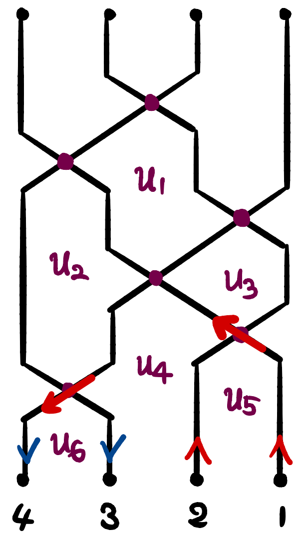
e.g> $\mathring{i} = (2.1.3.2.3.1)$ $\quad \mathbb{R}^6 = \mathbb{R}\langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$



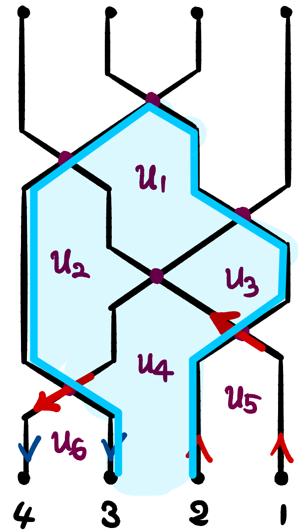
$G(\mathring{i}, 1)$



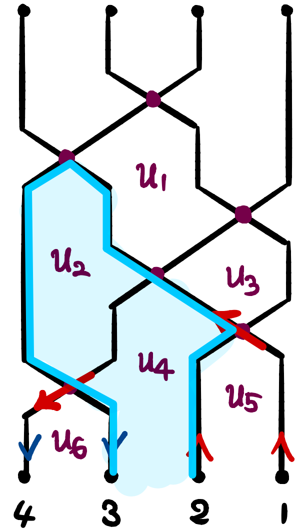
$u_5 \geq 0$



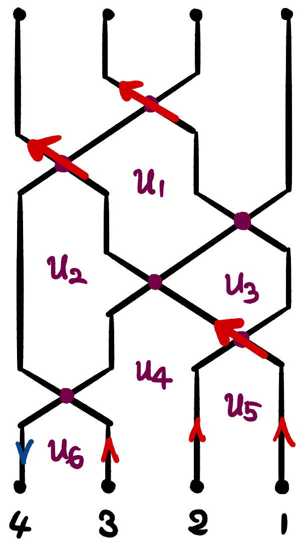
$G(\mathring{i}, 2)$



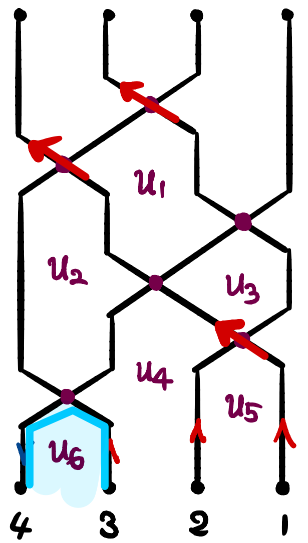
$u_1 + u_2 + u_3 + u_4 \geq 0$



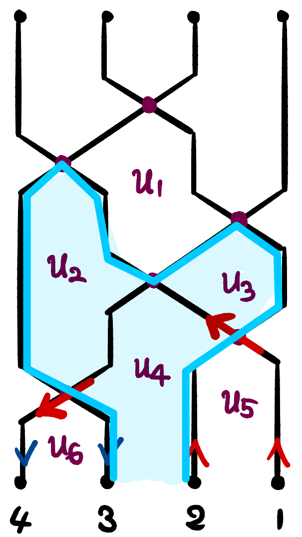
$u_2 + u_4 \geq 0$



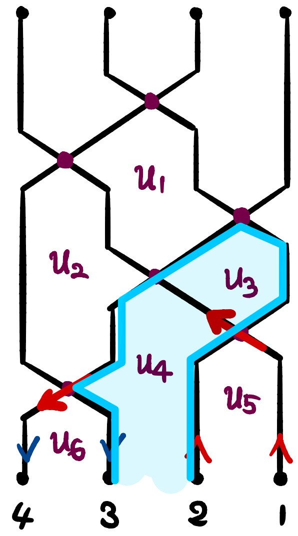
$G(\mathring{i}, 3)$



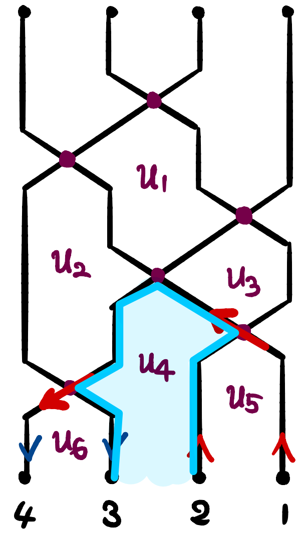
$u_6 \geq 0$



$u_2 + u_3 + u_4 \geq 0$



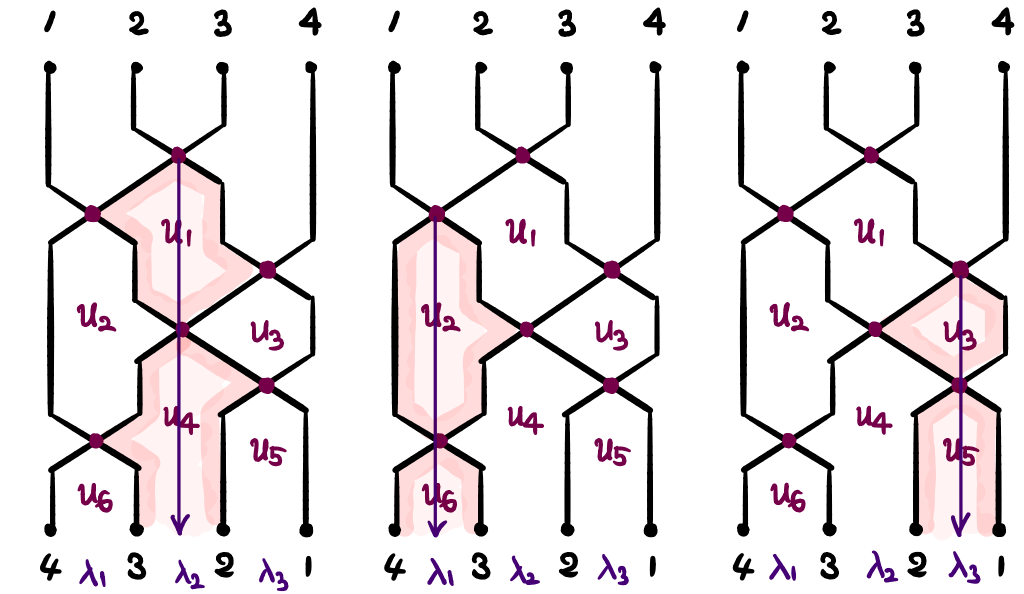
$u_3 + u_4 \geq 0$



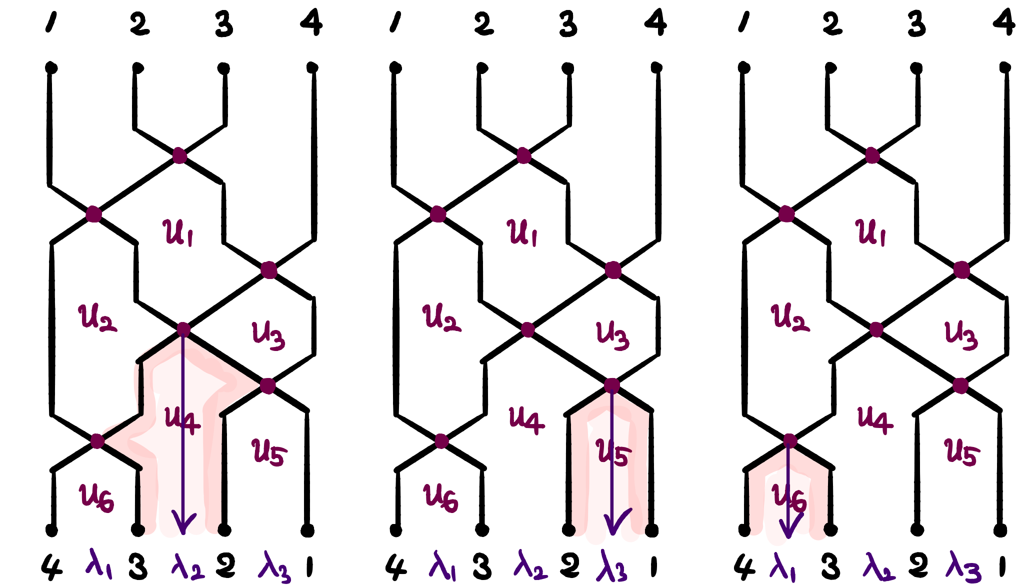
$u_4 \geq 0$

Littelmann's λ-cone

e.g> $\mathbf{i} = (2.1.3.2.3.1)$



$\lambda_2 - u_1 - u_4 \geq 0$ $\quad$ $\lambda_1 - u_2 - u_6 \geq 0$ $\quad$ $\lambda_3 - u_3 - u_5 \geq 0$



$\lambda_2 - u_4 \geq 0$ $\quad$ $\lambda_3 - u_5 \geq 0$ $\quad$ $\lambda_1 - u_6 \geq 0$

String Polytope

The string polytope is defined by

(String Cone) $\cap$ (λ Cone)

e.g> $\mathbf{i} = (2.1.3.2.3.1)$

$\Delta_{\mathbf{i}} \subseteq \mathbb{R}^6$ is defined by

$$
\begin{array}{cccccccl}
 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & \\
\left\{\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\right. & 1 & 1 & 1 & 1 & & & \geq 0 \\
 & & 1 & 1 & 1 & & & \geq 0 \\
 & & 1 & & 1 & & & \geq 0 \\
 & & & 1 & 1 & & & \geq 0 \\
 & & & & 1 & & & \geq 0 \\
 & & & & & 1 & & \geq 0 \\
 & & & & & & 1 & \geq 0 \\
\left\{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right. & -1 & & & -1 & & & \geq -\lambda_2 \\
 & & -1 & & & & -1 & \geq -\lambda_1 \\
 & & & -1 & & -1 & & \geq -\lambda_3 \\
 & & & & -1 & & & \geq -\lambda_2 \\
 & & & & & -1 & & \geq -\lambda_3 \\
 & & & & & & -1 & \geq -\lambda_1
\end{array}
$$

String cone

λ cone

Monotone Lagrangian Tori in $\mathcal{F}\ell(1,2,\cdots,n-1;n)$

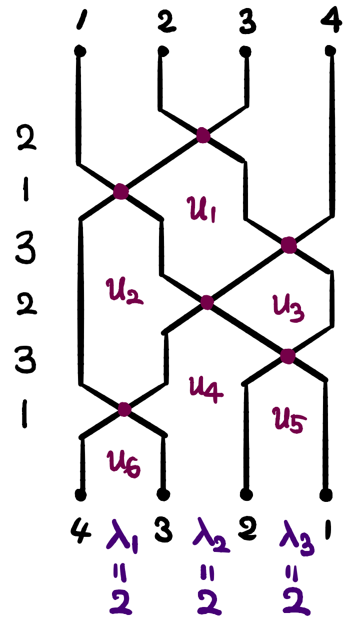
If we choose $\lambda = 2\varpi_1 + \cdots + 2\varpi_{n-1}$, then a monotone Lagrangian torus occurs at

(Bounded chamber var = 0) (Unbounded chamber var = 1)

e.g> $\mathring{i} = (2.1.3.2.3.1)$

A monotone Lagrangian torus is located at

$$\begin{cases} u_1 = u_2 = u_3 = 0 \\ u_4 = u_5 = u_6 = 1 \end{cases}$$



In sum, fix λ for $\mathcal{F}\ell(1,2,\cdots,n-1;n)$.

$\boxed{\text{For each choice of } \mathring{i}, \text{ there exists a monotone Lagrangian torus in } \mathcal{F}\ell(1,2,\cdots,n-1;n)}$

Key ingredient. Maslov index formula for gradient disk. (Y. Cho - K.)

The Number of String Polytopes

But, for a fixed n, there are only finitely many choices of reduced expressions of the longest element of S_n

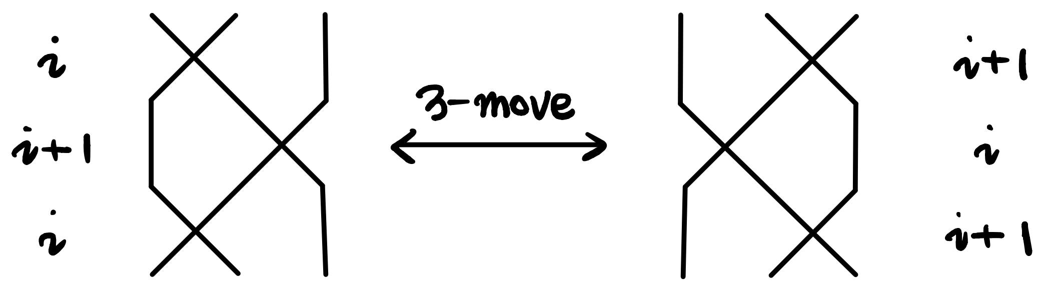
$\Rightarrow$ There are only finitely many string polytopes & toric degenerations for $\mathcal{F}\ell(1,2,\cdots,n-1;n)$

$\Rightarrow$ There are only finitely many monotone Lag. arising from a string polytope.

(Tits) Every pair of reduced expressions of the longest element of S_n is related by a finite sequence of braid moves (2-moves & 3-moves)

(i) (2-moves) $(i\ j) \longleftrightarrow (j\ i)$ $(|i-j|>1)$

(ii) (3-moves) $(i\ i{+}1\ i) \longleftrightarrow (i{+}1\ i\ i{+}1)$



Facts.

- Indeed, 2-move does not produce a new string polytope...
- Not every pair of distinct reduced expressions produce different polytopes

Cluster Algebras and Toric degenerations

Question How can we make infinitely many distinct monotone Lag. tori ?

(Gross – Hacking – Keel – Kontsevich, Fujita – Oya)

Toric degenerations of compactified $\mathcal{A}$-cluster variety can be described by Fock – Goncharov dual $\mathcal{X}$-cluster variety

In our case,

$$\boxed{\Delta : \text{string polytope} \xrightarrow[\text{combinatorial mutation}]{\mu_k} \Delta' : \text{generalized string polytope (Newton – Okounkov bodies)}}$$

(Anderson, Harada – Kaveh)

Produce toric degenerations and completely integrable systems.

We then obtain monotone Lagrangian tori

Mutations

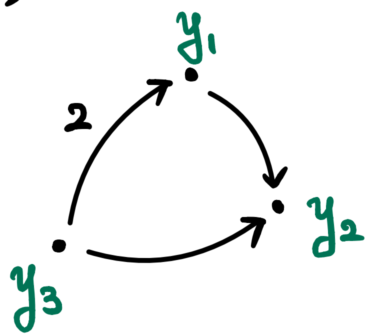
- **Seed**

$$\vec{y} = (\underbrace{y_1, y_2, \cdots, y_n}_{\text{cluster var.}}, \underbrace{y_{n+1}, \cdots, y_m}_{\text{frozen var.}})$$

$$B = \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix}}_{\text{skew-symmetric}}$$

- **Seed as a quiver**

e.g.>



$$B = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{array}$$

- **Mutations ($\mathcal{X}$-mutations)**

$$\mu_k(\vec{y}, B) = (\vec{y}', B') \quad (k = 1, 2, \cdots, n)$$

$$\vec{y}' = (y_1', y_2', \cdots, y_n', y_{n+1}', \cdots, y_m')$$

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j(1+y_k)^{-b_{kj}} & \text{if } j \neq k, b_{kj} \leq 0 \\ y_j(1+y_k^{-1})^{-b_{kj}} & \text{if } j \neq k, b_{kj} \geq 0 \end{cases}$$

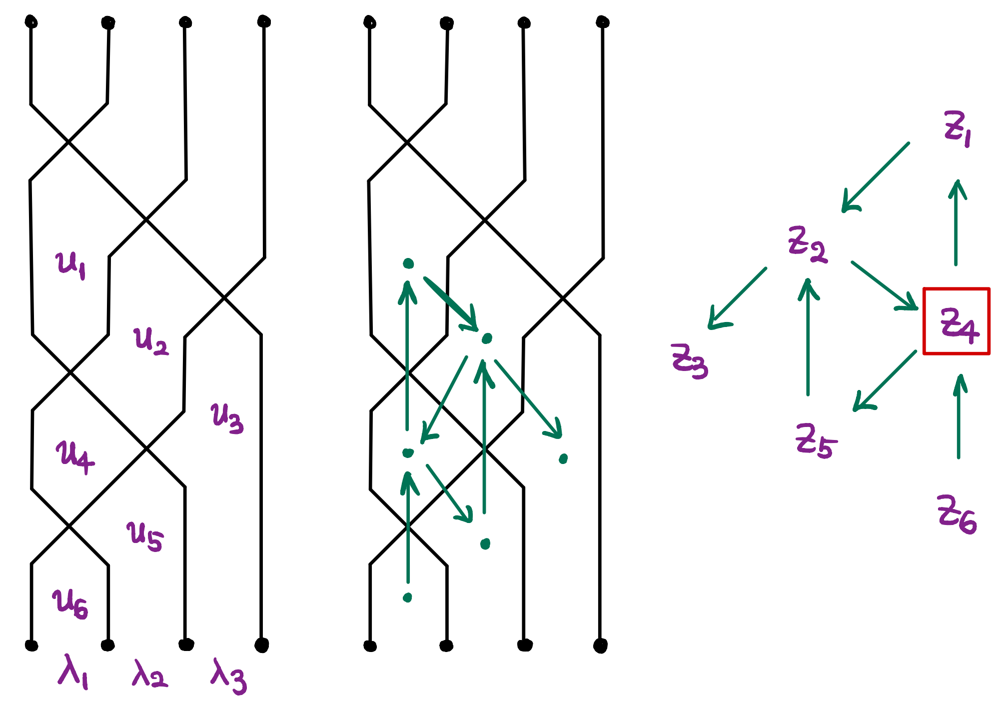
$$B' = (b_{ij}')_{m \times n}$$

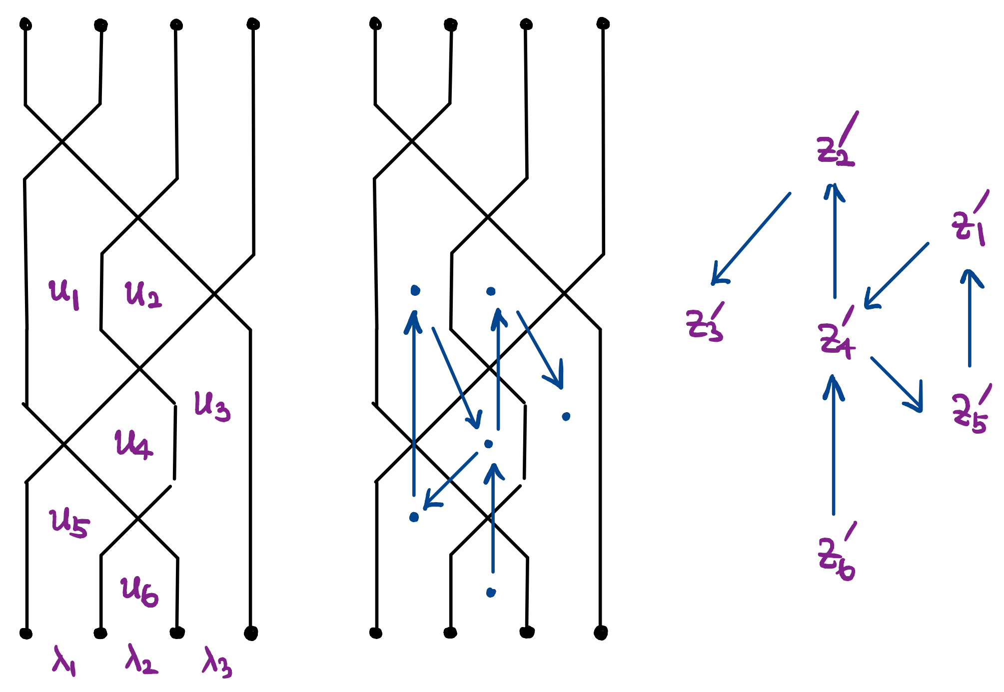
$$b_{ij}' = \begin{cases} -b_{ij} & (i = k \text{ or } j = k) \\ b_{ij} \pm b_{ik} b_{kj} & (b_{ik} \,\&\, b_{kj} \gtrless 0) \\ b_{ij} & (\text{otherwise}) \end{cases}$$

3-moves and Mutations

e.g.> $\mathbf{i} = (1,2,3,\underline{1,2,1})$

$\mu_4(\mathbf{i}) = (1,2,3,\underline{2,1,2}) = \mathbf{i}'$





$$W_{\mathbf{i}} = z_1z_2z_3 + z_2z_3 + z_3 + z_4z_5 + z_5 + z_6 + \frac{1}{z_1z_4z_6}T^{\lambda_1} + \frac{1}{z_2z_5}T^{\lambda_2} + \frac{1}{z_3}T^{\lambda_3} + \frac{1}{z_4z_6}T^{\lambda_1} + \frac{1}{z_5}T^{\lambda_2} + \frac{1}{z_6}T^{\lambda_1}$$

	1	2	3	4	5	6
1	0	1	0	-1	0	0
2	-1	0	1	1	-1	0
3	0	-1	0	0	0	0
4	1	-1	0	0	1	-1
5	0	1	0	-1	0	0
6	0	0	0	1	0	0

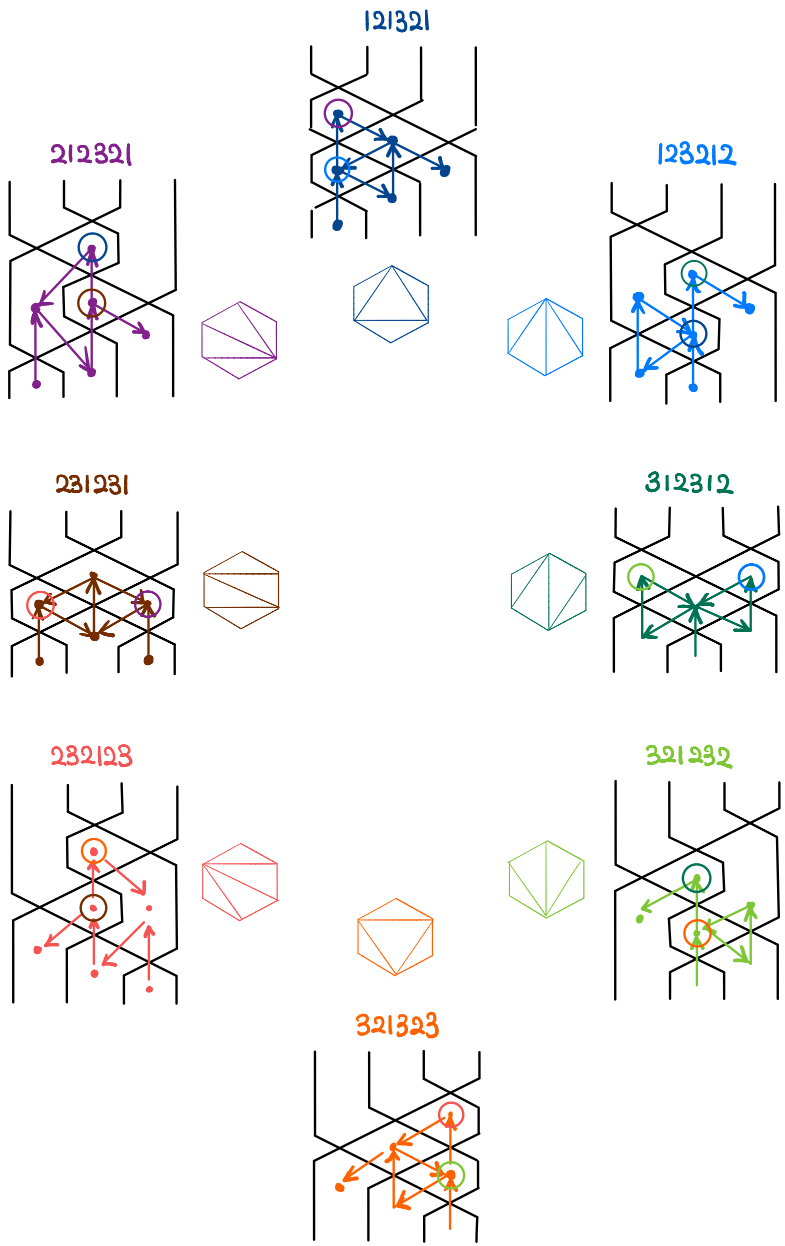
mutation

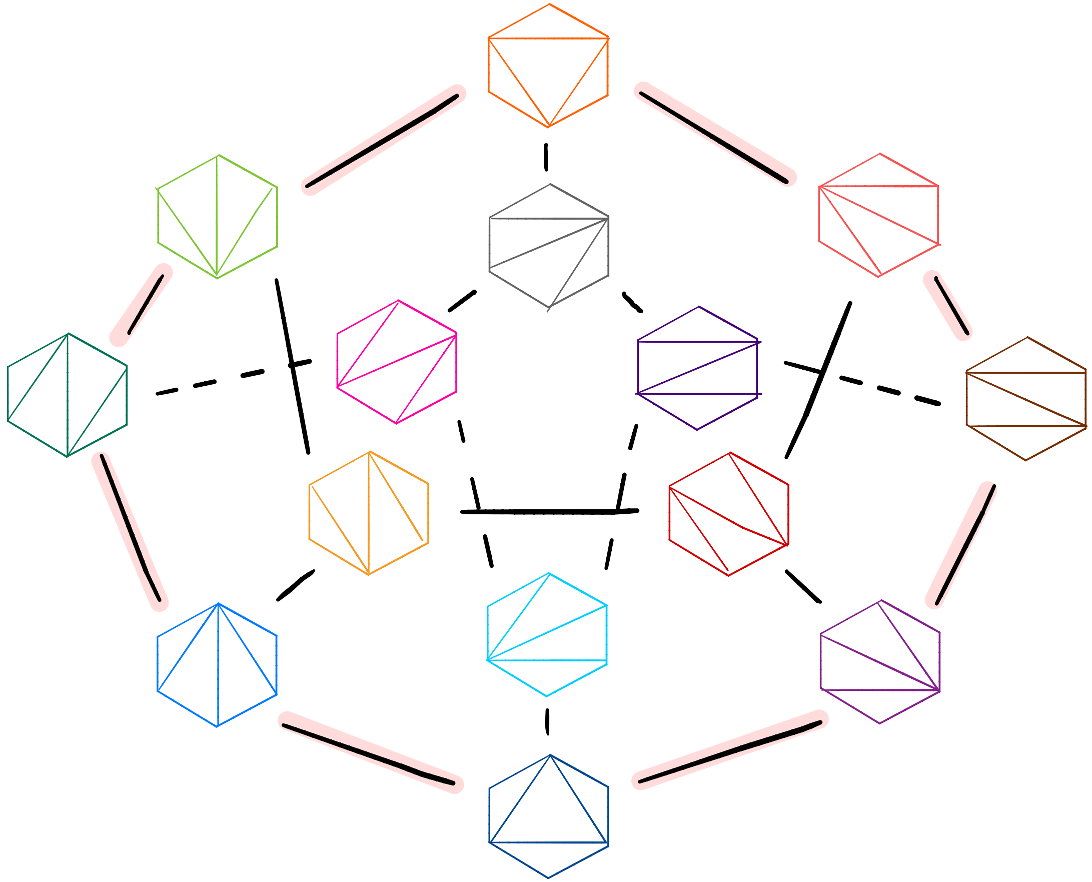
$$\mu_4(W_{\mathbf{i}}) = z_1'z_2'z_3'z_4' + z_2'z_3'z_4' + z_2'z_3' + z_3' + z_5' + z_6' + z_4'z_6' + \frac{1}{z_1'z_6'}T^{\lambda_1} + \frac{1}{z_2'z_4'z_5'}T^{\lambda_2} + \frac{1}{z_3'}T^{\lambda_3} + \frac{1}{z_6'}T^{\lambda_1} + \frac{1}{z_5'}T^{\lambda_2} + \frac{1}{z_4'z_5'}T^{\lambda_2}$$

$$\mu_4 : \begin{cases} z_1 = z_1'(1+z_4'^{-1})^{-1}, & z_2 = z_2'(1+z_4'), & z_3 = z_3' \\ z_4 = z_4'^{-1}, & z_5 = z_5'(1+z_4'^{-1})^{-1} & z_6 = z_6'(1+z_4') \end{cases}$$

Wiring Diagram and Quiver Mutations

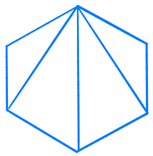
e.g> $n=4$ finite type (A_3) → has only finitely many seeds

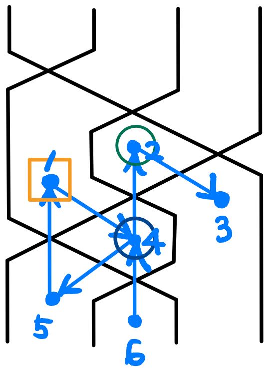




New Monotone Torus

Produce a new monotone Lagrangian torus different from one from string polytope.





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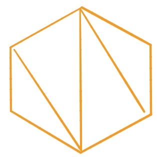
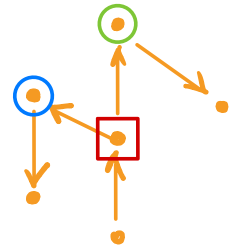
Exchange

	1	2	3	4	5	6
1				1	-1	
2			1	-1		
3		-1				
4	-1	1			1	-1
5	1			-1		
6				1		





Mutation $z_1 = z_1'^{-1},\ z_4 = z_4'(1+z_1'^{-1})^{-1},\ z_5 = z_5'(1+z_1')$

$$W = z_1z_2z_3z_4 + z_2z_3z_4 + z_2z_3 + z_3 + z_4z_5 + z_5 + z_6 + z_1^{-1}z_5^{-1}T^{\lambda_1} + z_2^{-1}z_4^{-1}z_6^{-1}T^{\lambda_2} + z_3^{-1}T^{\lambda_3} + z_4^{-1}z_6^{-1}T^{\lambda_2} + z_5^{-1}T^{\lambda_1} + z_6^{-1}T^{\lambda_2}$$

$$W = z_2'z_3'z_4' + z_2'z_3' + z_3' + z_1'z_4'z_5' + z_5' + z_1'z_5' + z_6' + z_5'^{-1}T^{\lambda_1} + z_1'^{-1}z_2'^{-1}z_4'^{-1}z_6'^{-1}T^{\lambda_2} + z_2'^{-1}z_4'^{-1}z_6'^{-1}T^{\lambda_2} + z_3'^{-1}T^{\lambda_3} + z_1'^{-1}z_4'^{-1}z_6'^{-1}T^{\lambda_2} + z_4'^{-1}z_6'^{-1}T^{\lambda_2} + z_6'^{-1}T^{\lambda_2}$$

↑ superpotential.

a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	1	2	3	4	5	6
	1	1	1	1		
		1	1	1		
		1	1			
			1			
				1	1	
					1	
						1

	a_0	a_1	a_2	a_3	a_4	a_5	a_6
	0	1	2	3	4	5	6
λ_1		-1				-1	
λ_2			-1		-1		-1
λ_3				-1			
λ_2					-1		-1
λ_1						-1	
λ_2							-1

↓ tropicalization

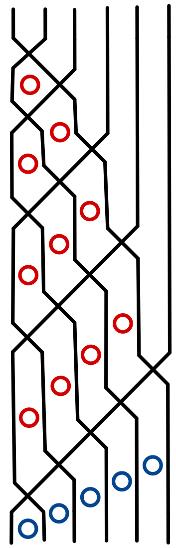
a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	1	2	3	4	5	6
		1	1	1		
		1	1			
			1			
	1			1	1	
	1				1	
					1	
						1

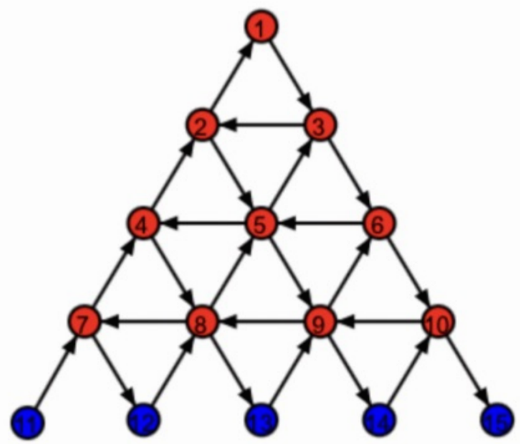
	a_0	a_1	a_2	a_3	a_4	a_5	a_6
	0	1	2	3	4	5	6
λ_1						-1	
λ_2		-1	-1		-1		-1
λ_2			-1		-1		-1
λ_3				-1			
λ_2		-1			-1		-1
λ_2					-1		-1
λ_2							-1

Infinitely Many Monotone Tori

If $n \geq 6$, then there are infinitely many toric degenerations arising from quiver mutations

e.g.> $\mathbf{i} = (1.2.1.3.2.1.4.3.2.1.5.4.3.2.1) \rightsquigarrow \Delta_{\mathbf{i}}$.





$\rightsquigarrow$ There are infinitely many monotone Lagrangian tori.

Question How can we distinguish them?

Indeed, not every one is different and some of them are same

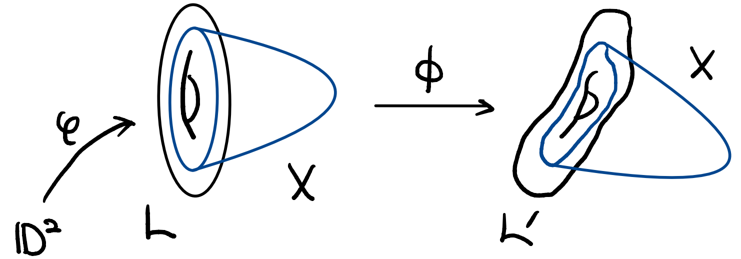
Question Are given pair of monotone Lag. tori same or different.

Counting Invariants

Lemma (Eliashberg – Polterovich)

Suppose that L and L' are monotone.
If L and L' are Hamiltonian isotopic, that is, $\phi(L) = L'$, then

$$n(L, \beta) = n(L', \phi_*\beta)$$

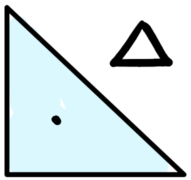
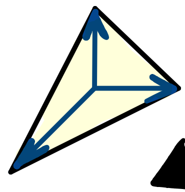


Suppose that two monotone L and L' are Hamiltonian isotopic.

$\Rightarrow$ Two disk potentials W_L and $W_{L'}$ are same (up to coordinate change)

$\Rightarrow$ Newton polytopes $\blacktriangle_L^\vee$ of W_L (resp. $\blacktriangle_{L'}^\vee$ of $W_{L'}$) are unimodular equiv.

$(SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n)$

e.g>  Δ $\rightsquigarrow$ $W = y_1 + y_2 + \dfrac{1}{y_1 y_2}$ $\rightsquigarrow$  $\blacktriangle^\vee$

Thus, if $\blacktriangle_L^\vee$ and $\blacktriangle_{L'}^\vee$ are not unimodular equiv., then L and L' are different.

But. disk potential functions W_L and $W_{L'}$ are not known

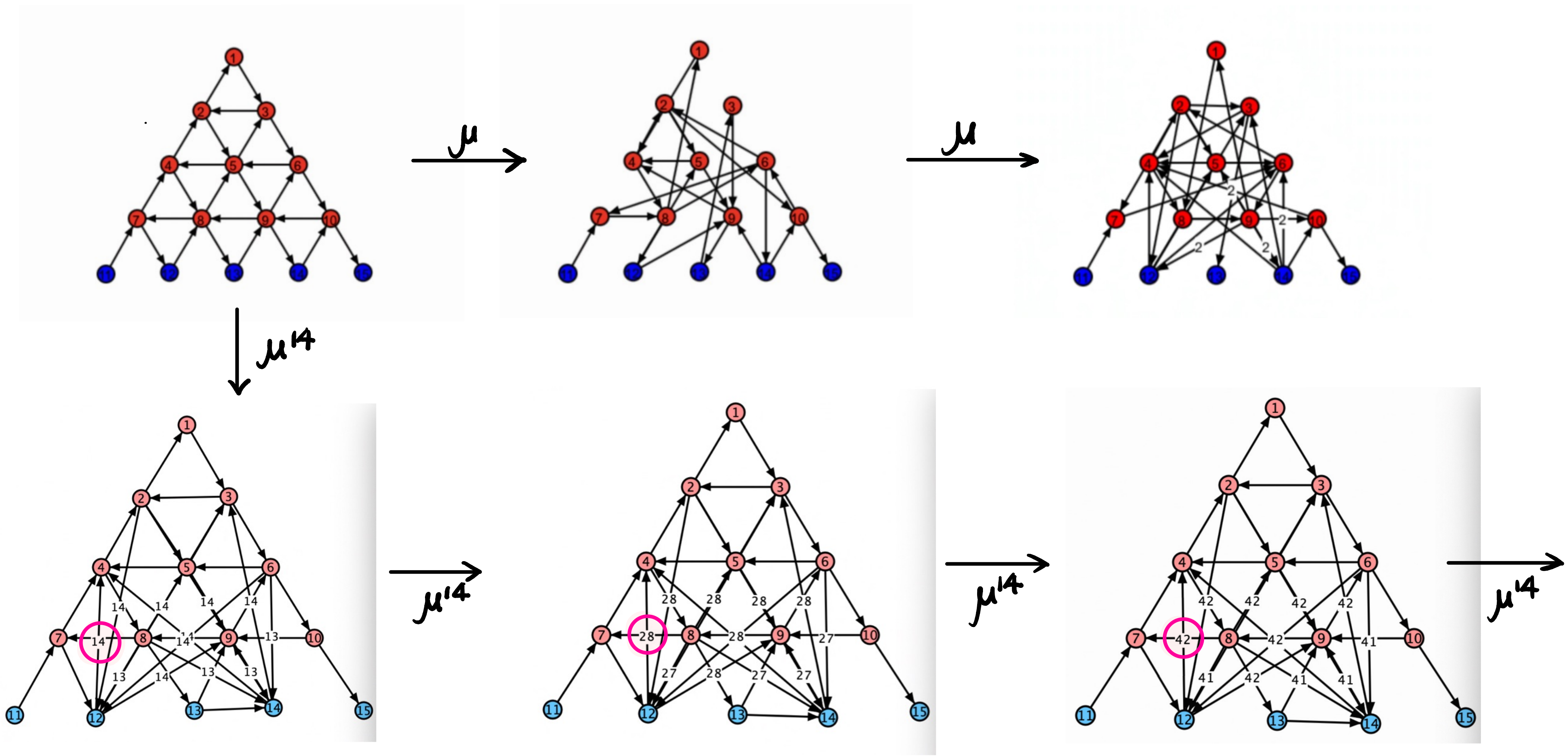
cf> (Y. Cho – K – Lee – Park) $W_{\mathbf{i}}$ is computed if $\mathbf{i}$ is "close" to $\mathbf{i}_{std} = (1\,2\,1\,3\,2\,1\cdots)$

Lemma $\blacktriangle_L^\vee \supseteq \blacktriangle_{Comb}^\vee$.

14 x 6 Pseudo-Periodicity

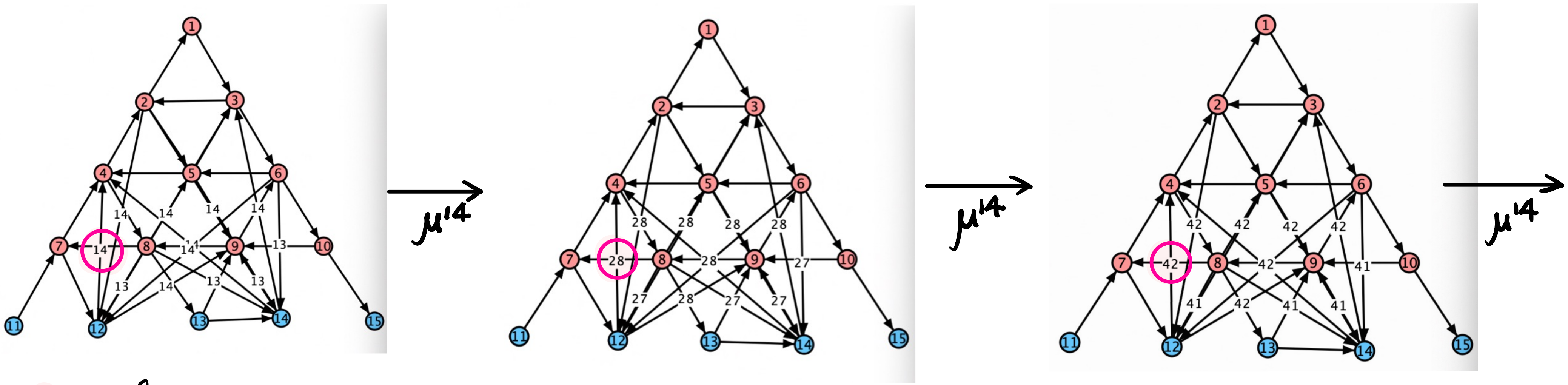
Let us control asymptotic behavior of particular sequence of mutations

e.g> $(n=6)$ $\mu := \mu_4 \circ \mu_8 \circ \mu_9 \circ \mu_6 \circ \mu_3 \circ \mu_2$. $\mu^n := \mu^{n-1} \circ \mu$



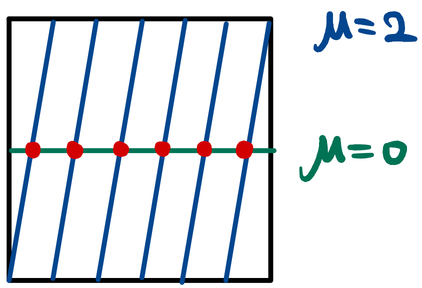
grows linearly.

Meaning of Multiplicities of Arrows

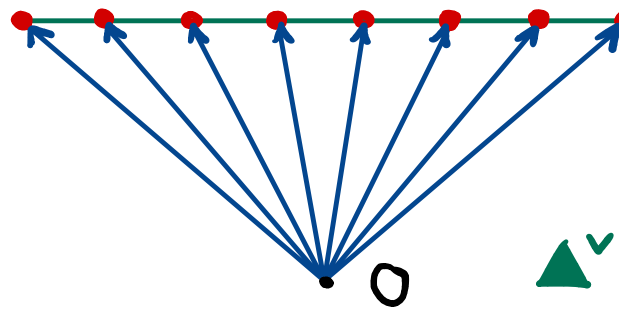


multiplicity

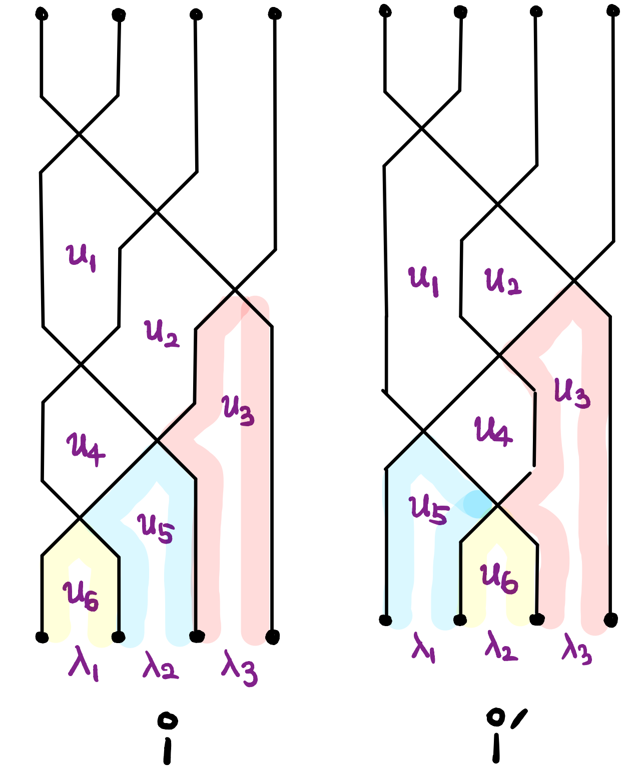
(1) Wall-crossing

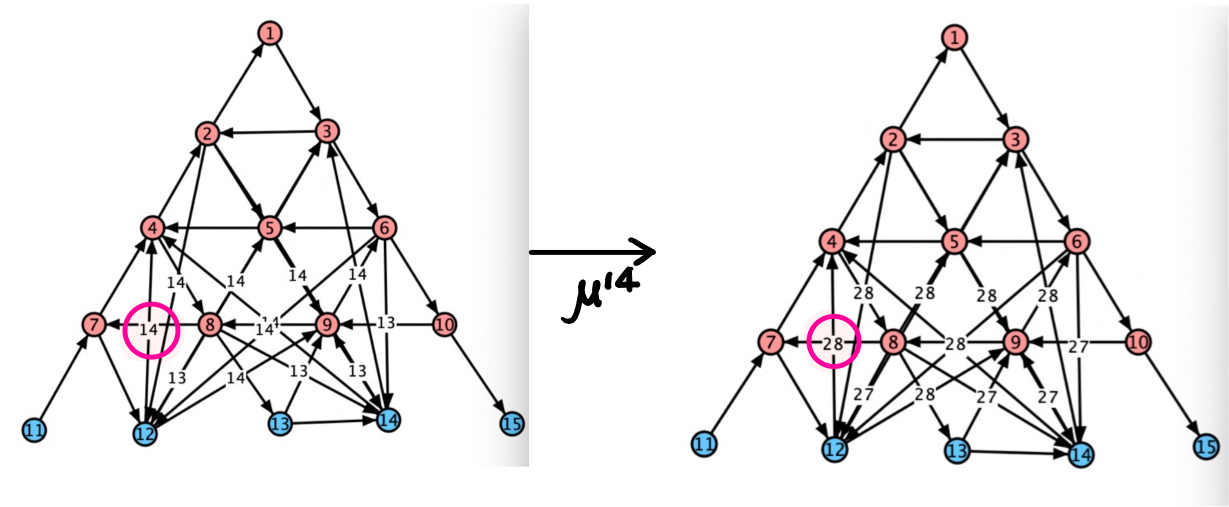


(2) Newton polytope



Pivot Terms in Potentials





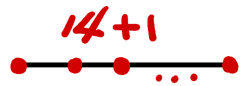
$$W_{\mathbf{i}} = z_3 + z_5 + z_6 + \underline{W_{\mathbf{i}}}$$

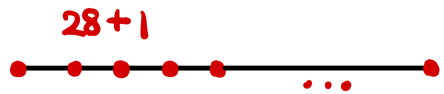
$$W_{\mathbf{i}'} = z_3' + z_5' + z_6' + \underline{W_{\mathbf{i}'}}$$

W always contains z_{frozen} term even after performing any sequence of mutations.

Applying the mutation rule to $z_{\text{frozen}=12}$. we conclude

$$\begin{cases} \mu^{14}(W_{in}) \text{ contains } z_{12}(1+z_4)^{14} \\ \mu^{28}(W_{in}) \text{ contains } z_{12}(1+z_4)^{28} \\ \vdots \end{cases} \Rightarrow \quad \begin{array}{l} \blacktriangle^{\vee}_{\mu^{14}} \text{ contains} \\ \blacktriangle^{\vee}_{\mu^{28}} \text{ contains} \end{array}$$





The number of lattice pts of $\blacktriangle^{\vee}_{\mu^{14n}} \geq 14n$

Thank you !