Torelli Theorem of ALH^* Gravitational Instantons

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Outline of the Talk

- Gravitational Instantons and their Classifications
- ALH^* Gravitational Instantons
- From SYZ Conjecture to Gravitational Instantons
- Torelli Theorem of ALH^* Gravitational Instantons
- Compactification Result of ALH^* Gravitational Instantons

Gravitaional Instantons

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- Mathematically, gravitational instantons are non-compact complete hyperKähler 4-manifolds with L^2 curvature.

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- Mathematically, gravitational instantons are non-compact complete hyperKähler 4-manifolds with L^2 curvature.
- A 4-manifold X is hyperKähler if it admits
  1. a Kähler form ω and
  2. a holomorphic volume form Ω

  s.t. the complex Monge-Ampere equation $2\omega^2 = \Omega \wedge \bar{\Omega}$ holds.
- They arises as the blow up limits of hyperKähler metrics. Ref: Foscolo'16, HSVZ '18, SZ'19, CVZ '19.

Classification of Gravitational Instantons

Classification of Gravitational Instantons I

- From the volume growth, people found gravitational instantons of type $ALE(r^4)$,$ALF(r^3)$, $ALG(r^2)$, $ALH(r)$.
- ALE stands for $a$symptotically locally Eucledean, F for flat and the rest by induction.

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- ALE stands for *a*symptotically locally Eucledean, F for flat and the rest by induction.
- (Hein '12) constructed two new gravitational instantons ALG^*, ALH^* with volume growth $r^2, r^{4/3}$. The former has a different curvature decay from ALG.
- (Sun-Zhang '21) The above are the exhaustive list of gravitational instantons.

Rational Elliptic Surfaces

- **Rational elliptic surfaces** (RES) are projective rational surface with an elliptic fibration structure.
- They are always realized as blow up of the base points (9 point, possibly infinitely near) of a cubic pencil in $\mathbb{P}^2$.

Rational Elliptic Surfaces

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- They are always realized as blow up of the base points (9 point, possibly infinitely near) of a cubic pencil in $\mathbb{P}^2$.
- (Hein '12) ALH, ALG, ALG^*, ALH^* gravitational instantons can be realized on the complement of a fibre in a rational elliptic surface, with the fibre is of type I_0, finite monodromy, I_k^* or I_k.

Classification of Gravitational Instantons II

- (Kroheimer '89) Torelli theorem for ALE and period domain. Two ALE gravitational instantons with the HK triples in the same cohomology class, the HK triples are the same.
- (Chen-Chen '15 '16) ALF, ALH case.

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Theorem (Collins-Jacob-L'21)

$(X_i, \omega_i, \Omega_i)$ ALH^* gravitaional instantons. If $f : X_2 \to X_1$ diffeo. s.t. $f^*[\omega_1] = [\omega_2], f^*[\Omega_1] = [\Omega_2]$. Then $\exists F : X_2 \to X_1$ s.t. $F^*\omega_1 = \omega_2, F^*\Omega_1 = \Omega_2$.

- (Chen-Viaclovsky-Zhang '21) ALG, ALG^* case.

Calabi-Yau Manifolds and SYZ Conjecture

Calabi-Yau Manifolds

- Calabi-Yau n-fold X
  =higher dimension analogue of elliptic curves.
  =complex manifold X with
  1. nowhere vanishing holomorphic n-form Ω
  2. d-closed non-degenerate positive $(1,1)$-form ω such that $\omega^n = c\Omega \wedge \bar{\Omega}$.

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- Examples:
  - degree 5 hypersurface in $\mathbb{P}^4$ (quintic 3-fold).
  - (Tian-Yau) complement of a smooth anti-canonical divisor in a Fano manifold

Strominger-Yau-Zaslow Conjecture

Conjecture (Strominger-Yau-Zaslow '96)

- *Calabi-Yau manifolds admit* **special Lagrangian** *torus fibration near large complex structure limit.*
- *Mirror Calabi-Yau are constructed by dual torus fibration.*

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1. (Harvey-Lawson '82) A submanifold L in X is special Lagrangian if $\omega|_L = 0$, $\mathrm{Im}\Omega|_L = 0$.
2. (Duistermaat '80) Any compact fibre of a Lagrangian fibration is topologically a torus.

Why SYZ Conjecture?

The SYZ conjecture is important in various aspects:

- It gives a geometric description of Calabi-Yau manifolds.
- It provides a recipe to construct the mirror $\check{X}$.
- (Leung-Yau-Zaslow, FLTZ, CPU, FHLY, CHL,...)
  It gives a guidance of how branes mirror to each other in the homological mirror symmetry.
- It motivates the method of family Floer mirror and Gross-Siebert program in mirror symmetry.

ALH^* Gravitational Instantons

Calabi Model and ALH^* Gravitional Instantons

- D elliptic curve, L line bundle on D w/ $\deg L = k > 0$.
- $Y_{\mathcal{C}}$=neighborhood of zero section of L, $\pi : Y_{\mathcal{C}} \to D$.
- $X_{\mathcal{C}} = Y_{\mathcal{C}} \backslash D$.
- $\Omega_{\mathcal{C}} = \frac{dw}{w} \wedge \pi^* \Omega_D$.
- $\omega_{\mathcal{C}} = \sqrt{-1}\frac{2}{3}\partial\bar{\partial}(-\log|\xi|_h^2)^{\frac{3}{2}}$, where $\omega_D = \sqrt{-1}\partial\bar{\partial}h$.
- Set $l_0 = \big(-\log|\xi|_h^2\big)^{\frac{1}{4}}$. Then
  1. $|\nabla^k Rm| \le C_k l_0^{-(k+2)}$ has good control and
  2. $C_\iota^{-1} l_0^{-1} \le \mathrm{inj} \le C_\iota l_0^{-1}$ degenerates.
- (Sun-Zhang '21) ALH^* gravitational instantons are exponentially decay to the Calabi model at infinity.

Examples of ALH^* Gravitational Instantons

- (Tian-Yau '90) Y weak del Pezzo surface, D smooth anti-canonical, then $X = Y \setminus D$ is ALH^*.
- (Hein '12) $\check{Y}$ rational elliptic surface, $\check{D}$ I_k-fibre, then $\check{X} = \check{Y} \setminus \check{D}$ is ALH^*.

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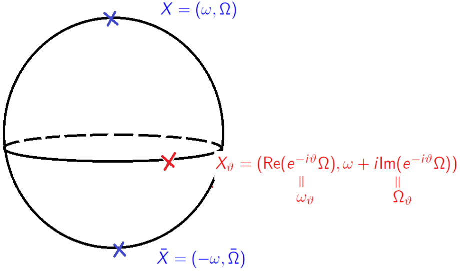
- (Hein '12) $\check{Y}$ rational elliptic surface, $\check{D}$ I_k-fibre, then $\check{X} = \check{Y} \setminus \check{D}$ is ALH^*.
- (Hein '12) They have the same asymptotics of *inj*, *Rm*, volume growth,... are all the same.
- Are these two examples related by HK rotation?

HyperKähler Rotation

The hyperKähler triple (ω, Ω) induces an S^2-family of complex structures on the underlying space of X.



Then holomorphic curves in X $\Leftrightarrow$ special Lagrangians in X_ϑ.

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Theorem (CJL '19)

Suitable HK rotation $\check{X}$ of X can be compactified to a rational elliptic surface $\check{Y}$.

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Theorem (CJL '20)

There exists a $\mathbb{R}$-family of HK metrics on $\check{X}$ with $\mathbb{Z} \subseteq \mathbb{R}$ corresponds Hein's metric.

Discovery of "non-standard semi-flat metric", new ansatz for HK metrics.

Uniformization of ALH^* Gravitational Instantons

Theorem (CJL' 21, HSVZ '21)

Up to HK rotation, any ALH^* gravitational instanton X can be compactified to a rational elliptic surface.

- Special Lagrangian tori fibraton in $X_{\mathcal{C}}$.
- Lagrangian mean curvature flow the ansatz fibration to a special Lagrangian near infinity of X.
- $\check{X}$ HK rotation of X can be compactified to a compact complex surface $\check{Y}$.
- Enrique-Kodaira classificaton $\Rightarrow$ $\check{Y}$ is RES.

(Persson) Classification of singular fibres in RES $\Rightarrow 1 \leq k \leq 9$.
10 different diffeomorphism types of ALH^* gravitational instantons.

Lagrangian Mean Curvature Flow

- Let L be a graded Lagrangian submanifold in X, i.e., $\exists$ the phase $\theta : L \to \mathbb{R}$ is the function such that

$$\Omega|_L = e^{i\theta} vol_L.$$

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$$\Omega|_L = e^{i\theta} vol_L.$$

  L is a special Lagrangian if θ is a constant.
- The mean curvature $\vec{H} = J\nabla\theta$ and the mean curvature flow is given by evolving family of immersions $F_t : L \to X$ with

$$\frac{\partial}{\partial t} F_t = \vec{H}.$$

- (Smoczyk) Maslov zero Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds.
- Smooth Convergent Limit of LMCF gives Special Lagrangians.

The Torelli Theorem

Warm-Up: Torelli Theorem for K3 Surfaces

- K3 surfaces are simply connected, compact surface with trivial canonical bundles.
- $\mathbb{L}_{K3} := H^2(K3) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $[\Omega] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C})$ determines the complex structure if
  $[\Omega] \wedge [\Omega] = 0, [\Omega] \wedge [\bar{\Omega}] > 0.$
- (Yau '76) Unique Ricci-flat metric in each Kähler class.

We will follow the same idea to prove the Torelli theorem of ALH^* gravitational instantons.

Torelli Theorem for Log CY Surfaces

Theorem (Gross-Hacking-Keel '15)

$(\check{Y}_i, \check{D})$ log Calabi-Yau surfaces w/ isom. $\mu : Pic(\check{Y}_1) \to Pic(\check{Y}_2)$,

1. *μ preserves the periods $\phi_i \in Hom(\check{D}^\perp, Pic^0(\check{D}))$,*

$$\phi_i : L \mapsto L|_{\check{D}}.$$

2. *μ preserves (-2)-curve classes*
3. *μ sends one Kähler class to a Kähler class.*

Then $\mu = F^*$, w/ isomoprhism of pairs $F : (\check{Y}_2, \check{D}) \to (\check{Y}_1, \check{D})$.

Without the 3rd assumption, F^* may be differed by an element in the Weyl group.

Proof of Torelli theorem for ALH^* Gravitational Instantons

- Construct an isometry of lattices $\mathrm{Pic}(\check{Y}_1) \to \mathrm{Pic}(\check{Y}_2)$ from f^*.
  1. Suitable homotopic modification of f to preserve sections.
  2. Computation of $MCG(X_{\mathcal{C}})$.
- GHK'15 $\Rightarrow \exists F : (\check{Y}_2, \check{D}) \cong (\check{Y}_1, \check{D})$.
  1. (Looijenga) $\phi_i = \exp\left(2\pi i \int \check{\Omega}\right)$.
  2. Discrepancy of Kähler classes on $\check{X}$ and $\check{Y}$.

Theorem (CJL '20)

HK metrics on $\check{X}$ with non-standard semi-flat metrics asymptotics and the same cohomology class are differed by translation of sections.

Replace F by composition w/ translation of sections.

Period Domain of Gravitational Instantons

- It still remains to ask given cohomology class of $[\omega], [\Omega]$, is there a gravitational instantons to realize the triple for the case of ALG, ALG^*, ALH^*?
- Obvious obstruction is $[\omega], [\Omega]$ can not simultaneously vanish on (-2)-classes.
- Chen-Viaclovsky-Zhang '21 conjectured no other obstruction.

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Theorem (Lee-Lin, work in progress)

Period doemain for ALG, ALG^*, ALH^* gravitational instantons.

Applications

Monodromy Action on the Fibration

- Recall that given a primitive element in $H_1(D, \mathbb{Z})$, there is a corresponding SYZ fibration in $X = Y \setminus D$.
- (Hacking-Keating '21) study the symplectmorphism of Looijenga pairs. Conjecture that the onodromy of the moduli of paris (Y, D) sends Lagrangian fibration to Lagrangian fibraiton.

Theorem (Lau-Lee-L.- '22)

Given a monodromy action on $H_2(X, \mathbb{Z})$, there exists an isometry of X realizing the monodromy and sending one SYZ fibration to another.

Maybe there is some application of lattice theory on this geometry?

Uniquess of Tian-Yau Metrics

Theorem (Collins-Jacob-L.- '21)

Let Y be a weak del Pezzo surface, D be a smooth anti-canonical divisor and $X = Y \setminus D$. Fix a meromorphic 2-form Ω on Y with a simple pole along D. For each Kähler class of X, there exists a unique hyperKähler metric ω in the given Kähler class with L^2 curvature.

This partially answers a question of Tian-Yau but can the L^2-condition be removed?

Compactification to Weak Del Pezzo Surfaces

Consider the HK rotation map

$$\Psi : \mathcal{M}_k \to \check{\mathcal{M}}_k$$
$$((Y, D), \mu, c, [\omega], \gamma) \mapsto ((\check{Y}, \check{D}), \check{\mu}, [\check{\omega}], \alpha)$$

- $\mathrm{Im}\Psi$ is open.
  construct log CY pairs with perturbed periods.
- $\mathrm{Im}\Psi$ is closed.
  Ψ continuous, some structure of moduli space of log CY pairs.

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- $\mathrm{Im}\Psi$ is closed.
  Ψ continuous, some structure of moduli space of log CY pairs.
- Ψ is surjective and Torelli theorem $\Rightarrow$ X compactified to Y.

Theorem (HSVZ '21, CL'22)

Every ALH* gravitational instanton (up to HK rotaion) can be compactified to a weak del Pezzo surface.

The compactification result further motivates the question:

Question

Can ALG, ALG^*-gravitaional instantons compactified to algebraic surfaces other than rational elliptic surfaces after suitable HK rotations?

Theorem (Collins-L.-, in progress)

True for gravitational instantons of second Betti number 5.

This has further applications in mirror symmetry: for instance how Gross-Hacking-Keel mirror symmetry is compatible with SYZ mirror symmetry, computation of certain local open GW invariants, how SYZ fibration detects the superpotentials...etc.

THANK YOU!