Effective Galois descent for motives: the K3 case

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June 27, 2022

Elliptic curves

This story, like every good story in algebraic geometry, begins with elliptic curves.

In fact, I'll do you one better. Let's start with a specific complex elliptic curve, namely

$$E/\mathbb{C} : y^2 = x^3 + \pi.$$

Question: What is the field of definition of E?

Precisely - what fields $K \subseteq \mathbb{C}$ can we find with an elliptic curve E_0/K such that $E_0 \otimes \mathbb{C} \xrightarrow{\sim} E$?

Well clearly we can descend E from $\mathbb{C}$ to $\mathbb{R}$. We can do better than that and further descend from $\mathbb{R}$ to the field $\mathbb{Q}(\pi)$. Hrm, is that the best we can do?

To address this, let's jump into a different context.

Some funny fields

We begin by recalling the *finite fields* $\mathbb{F}_p$: $\{0, 1, 2, \ldots, p-1\}$ and their extensions $\mathbb{F}_q$, where $q = p^n$.
From these we construct the *global function fields* $\mathbb{F}_q(T)$ and their extensions $\mathbb{F}_q(C)$, where $C \to \mathbb{P}^1_{\mathbb{F}_q}$ is a finite cover.
What do we know about these fields?

Theorem

The Galois group $\mathrm{Gal}(F_q/F_p)$ *is generated by the Frobenius map* $\mathrm{Fr} : x \mapsto x^p$*.*

Given an irreducible polynomial $\mathfrak{p}$ in $\mathbb{F}_p[T]$ we get a map of rings

$$\mathbb{F}_p[T] \longrightarrow \mathbb{F}_p[T]/\mathfrak{p} \xrightarrow{\sim} \mathbb{F}_q.$$

Counting points

Over a finite field $\mathbb{F}_q$ a variety will have a *finite* number of points, e.g. if we have E the elliptic curve $y^2 = x^3 + 1$ over $\mathbb{F}_5$ we have

$$E(\mathbb{F}_5) = \{(0,1),(0,4),(2,2),(2,3),(4,0),\infty\}, \quad \text{so} \quad \#E(\mathbb{F}_5) = 6.$$

Now let's consider some elliptic curves over $\mathbb{F}_q(T)$. For each of these we can *evaluate* at particular values of T and count points over the resulting finite field.

$$E_1 : y^2 = x^3 + x + T \qquad (18)$$
$$E_2 : y^2 = x^3 + T^6 \qquad (19)$$
$$E_3 : y^2 = x^3 + T^3 \qquad (20)$$

Here E_2 is constant, and E_3 is *isotrivial*, via

$$(x, y) \mapsto \left(Tx, (\sqrt{T})^3 y\right).$$

A Galois descent for elliptic curves

In fact these point counts are closely related to Galois groups! Each irreducible $\mathfrak{p}$ gives a Frobenius element $\mathrm{Fr}_{\mathfrak{p}}$ in $\mathrm{Gal}(\mathbb{F}_q(T)^{\mathrm{sep}}/\mathbb{F}_q(T))$.

Theorem

The quantity $a_{\mathfrak{p}} = \mathrm{Nm}(\mathfrak{p}) + 1 - \#E_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}})$ is the trace of $\mathrm{Fr}_{\mathfrak{p}}$ on the Tate module $T_\ell E$.

Now we can say something about Galois theory and descent here.

Theorem (ℓ-Galois descent for elliptic curves)

If the Galois action on the Tate module of E factors through a finite group, then E is isotrivial (i.e. constant after a finite extension).

Abelian varieties

There are many ways to generalise elliptic curves. One way is to study higher dimensional projective varieties with a group structure, i.e. *abelian varieties*. To these we can also associate a Tate module A and we have a descent theorem due to Grothendieck.

Theorem (ℓ-Galois descent for abelian varieties)

Let K/k be a regular extension. Let A/K be an abelian variety and $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\ell A)$ be the Galois representation on the Tate module. Assume $\rho(\mathrm{Gal}(K^{\mathrm{sep}}/k^{\mathrm{sep}}K)) = 1$. Then there exists A_0/k and an isogeny $A_0 \otimes K \to A$.

This isogeny can be taken to be an isomorphism in the following cases: (1) A is an elliptic curve, (2) K is characteristic 0, (3) A/K is *ordinary*.

What's in a ~~name~~ Tate module?

This is a complete and satisfying story for abelian varieties. Can we say anything for varieties in general?
The Tate module is a very useful vector space for abelian varieties. In fact we have an isomorphism $T_\ell A \cong H^1_{\text{ét}}(A, \mathbb{Z}_\ell)^\vee$ with the ℓ-adic étale cohomology. Cohomology is defined for any variety.

Example

Consider the surface over $\mathbb{F}_q(T)$ given by

$$\mathcal{E} : y^2 = x^3 - 27ux - 54v \qquad \text{over} \qquad C : v^2 = u^3 + T.$$

Then $H^1(\mathcal{E}, \mathbb{Z}_\ell) \cong H^1(C, \mathbb{Z}_\ell) \cong T_\ell C^\vee$.
In this case $H^2(\mathcal{E}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \,\mathrm{cl}(\mathrm{NS}\,\mathcal{E}) \oplus W$.

Again on $\mathcal{E}$ we can count points. (21)

Studying the surface $\mathcal{E}$

For $\mathcal{E}$ the cohomology is more complicated than the case of elliptic curves. We still have the trace $a_{\mathfrak{p}}$ of $\mathrm{Fr}_{\mathfrak{p}}$ on H^1, but this isn't so directly related to the point counts. To relate the point counts to the Galois action we'll use the following.

Proposition

The trace of $\mathrm{Fr}_{\mathfrak{p}}$ *on the transcendental lattice* W *is given by*

$$b_{\mathfrak{p}} = \#\mathcal{E}_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}}) - 1 + a_{\mathfrak{p}} - 12\,\mathrm{Nm}(\mathfrak{p}) + \mathrm{Nm}(\mathfrak{p})a_{\mathfrak{p}} - \mathrm{Nm}(\mathfrak{p})^2$$

Let's now take a look at these $b_{\mathfrak{p}}$ values. (22)
This pattern is again explained by $\mathcal{E}$ being isotrivial over $\mathbb{F}_q(T)$.
However, Grothendieck's theorem is very special, and I was convinced that I should be able to find examples of non-isotrivial surfaces $\mathcal{E}$ where you still had this finite Galois action.

The conjecture

Theorem (M.)

I was wrong.

In fact it is a "folklore" conjecture that this Galois property always implies descent. We in fact extend this to any *motive*.

Conjecture

Let K/k be a regular extension and H/K be a motive. Let $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(H_\ell)$ be the Galois representation on the ℓ-adic realisation. Assume $\rho(\mathrm{Gal}(K^{\mathrm{sep}}/k^{\mathrm{sep}}K)) = 1$. Then there exists a motive H_0/k^{sep} and an isomorphism $H_0 \otimes K^{\mathrm{sep}} \xrightarrow{\sim} H \otimes K^{\mathrm{sep}}$.

Grothendieck's theorem gives a proof of this conjecture for abelian varieties. Can we prove any other cases?

K3 surfaces

We began by studying elliptic curves, and generalised by looking at abelian varieties. Elliptic curves are the only curves with trivial *canonical bundle*. Among surfaces there are two classes with trivial canonical bundle, one is abelian surfaces.
The other is given by *K3 surfaces*.

Definition

A *K3 surface* over K is a smooth projective surface X/K with trivial canonical bundle ($\omega_X \cong \mathcal{O}_X$) and which is simply connected ($H^1(X, \mathcal{O}_X) = 0$).

Example

The *Fermat quartic* $x^4 + y^4 + z^4 + w^4 = 0$.

Some data on K3s

The most striking feature of a K3 surface X is its *Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

In particular, $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ is 22 dimensional. It comes equipped with a cup product pairing Q.
If $X/\mathbb{C}$ is a complex K3 surface, then $H^2(X, \mathbb{Z})$ is a weight 2 Hodge structure.

Main theorem

Theorem (M.)

Let K/k be a regular extension. Let X/K be a K3 surface and let $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(H^2_{\acute{e}t}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell))$ be the Galois representation on the second étale cohomology. Assume $\rho(\mathrm{Gal}(K^{\mathrm{sep}}/k^{\mathrm{sep}}K) = 1$ and either K is characteristic 0, or X/K is ordinary. Then there exists a K3 surface X_0/k^{sep} and an isomorphism $X_0 \otimes K^{\mathrm{sep}} \xrightarrow{\sim} X \otimes K^{\mathrm{sep}}$.

The strategy is to recognise that K3 surfaces are very closely related to abelian varieties.

Kuga-Satake abelian variety

There is a method by which we can associate an abelian variety to our K3 surface. This is the *Kuga-Satake construction*.
In the complex case $X/\mathbb{C}$ we take the lattice $L = H^2(X, \mathbb{Z})$ with its cup product pairing and form the *Clifford algebra*

$$\mathrm{Cl}(L) = \bigoplus_n L^{\otimes n} / \langle v \otimes v - Q(v, v) \rangle$$

Using the weight 2 Hodge structure on L, one may construct a weight 1 Hodge structure on $\mathrm{Cl}(L)$.

Theorem (Riemann)

There is an equivalence of categories between polarisable weight 1 Hodge structures and complex abelian varieties.

The main diagram

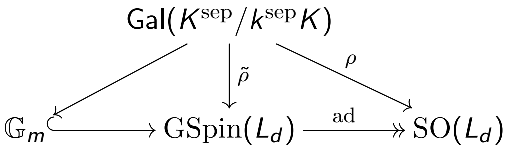
The above result gives a complex abelian variety, but we would like a construction that (a) works over arbitrary fields, and (b) can track information about Galois actions.
The key idea of extending and refining the Kuga-Satake construction above is the following diagram, due to Madapusi-Pera.

$$
\begin{array}{ccccc}
 & & \mathcal{S}(\mathrm{GSpin}(L_d)) & \longrightarrow & \mathcal{S}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\
 & & \downarrow & & \\
\tilde{M}_{2d,\gamma} & \longrightarrow & \mathcal{S}(\mathrm{SO}(L_d)) & &
\end{array}
$$

Here $\tilde{M}_{2d,\gamma}$ is a moduli space of γ-*oriented* K3 surfaces, and $\mathcal{S}(\cdot)$ is an integral model for a Shimura variety.

The proof idea

Beginning with a K3 surface $X \in \tilde{M}_{2d,\gamma}$ we can find an abelian variety $KS(X)$ above a point in $\mathcal{S}(\mathrm{GSpin}(L_d))$. We show the Galois representations are related by the following diagram:



In the theorem we assume the image under ρ is trivial, thus the image under $\tilde{\rho}$ is contained in $\mathbb{G}_m$. Then one can show that this is a root of the cyclotomic character, which is trivial since we are fixing k^{sep}. Thus $\tilde{\rho}$ is trivial and we can apply Grothendieck's theorem to $KS(X)$. Finally, we use our hypotheses on X to conclude that X itself descends.

Looking forward and back

Strangely, the proof was inspired by having first worked in the case of a Kummer surface. However, it transpired that the proof in that case was actually missing a representation-theoretic input. Here would be the key ingredient needed.

Conjecture

Let A/K be an abelian variety. Assume the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-action on $H^2(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is trivial. Then the action on $H^1(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is trivial.

Another step for the future would be to remove the hypotheses on K and X. This would mean that the Kuga-Satake variety would only descend *up to isogeny*. The question then is whether this isogeny is induced by an appropriate correspondence of K3 surfaces. In a different setting, this is known due to work of Yang.

Thanks for listening!

$E_1 : y^2 = x^3 + x + T$

Degree 2 irreducibles in $\mathbb{F}_7[T]$

Irreducible $\mathfrak{p}$	$\#E_{1,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{1,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{1,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$
T^2+1	48	T^2+2T+3	44	T^2+4T+6	45
T^2+2	60	T^2+2T+5	47	T^2+5T+2	46
T^2+4	63	T^2+3T+1	48	T^2+5T+3	44
T^2+T+3	58	T^2+3T+5	54	T^2+5T+5	47
T^2+T+4	49	T^2+3T+6	45	T^2+6T+3	58
T^2+T+6	38	T^2+4T+1	48	T^2+6T+4	49
T^2+2T+2	46	T^2+4T+5	54	T^2+6T+6	38

(Back to 4)

$E_2 : y^2 = x^3 + T^6$

Degree 2 irreducibles in $\mathbb{F}_7[T]$

Irreducible $\mathfrak{p}$	$\#E_{2,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{2,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{2,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$
T^2+1	48	T^2+2T+3	48	T^2+4T+6	48
T^2+2	48	T^2+2T+5	48	T^2+5T+2	48
T^2+4	48	T^2+3T+1	48	T^2+5T+3	48
T^2+T+3	48	T^2+3T+5	48	T^2+5T+5	48
T^2+T+4	48	T^2+3T+6	48	T^2+6T+3	48
T^2+T+6	48	T^2+4T+1	48	T^2+6T+4	48
T^2+2T+2	48	T^2+4T+5	48	T^2+6T+6	48

(Back to 4)

$E_3 : y^2 = x^3 + T^3$

Degree 2 irreducibles in $\mathbb{F}_7[T]$

Irreducible $\mathfrak{p}$	$\#E_{3,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{3,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#E_{3,\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$
T^2+1	48	T^2+2T+3	52	T^2+4T+6	52
T^2+2	48	T^2+2T+5	52	T^2+5T+2	48
T^2+4	48	T^2+3T+1	48	T^2+5T+3	52
T^2+T+3	52	T^2+3T+5	52	T^2+5T+5	52
T^2+T+4	48	T^2+3T+6	52	T^2+6T+3	52
T^2+T+6	52	T^2+4T+1	48	T^2+6T+4	48
T^2+2T+2	48	T^2+4T+5	52	T^2+6T+6	52

(Back to 4)

$\mathcal{E} : y^2 = x^3 - 27ux - 54v$ over $C : v^2 = u^3 + T$.

Degree 2 irreducibles in $\mathbb{F}_7[T]$

Irreducible $\mathfrak{p}$	$\#\mathcal{E}_\mathfrak{p}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#\mathcal{E}_\mathfrak{p}(\mathbb{F}_\mathfrak{p})$	Irreducible $\mathfrak{p}$	$\#\mathcal{E}_\mathfrak{p}(\mathbb{F}_\mathfrak{p})$
T^2+1	2988	T^2+2T+3	2340	T^2+4T+6	3090
T^2+2	2538	T^2+2T+5	3540	T^2+5T+2	2342
T^2+4	3738	T^2+3T+1	2792	T^2+5T+3	2340
T^2+T+3	2340	T^2+3T+5	3540	T^2+5T+5	3540
T^2+T+4	3542	T^2+3T+6	3090	T^2+6T+3	2340
T^2+T+6	3090	T^2+4T+1	2792	T^2+6T+4	3542
T^2+2T+2	2342	T^2+4T+5	3540	T^2+6T+6	3090

(Back to 7)

$\mathcal{E} : y^2 = x^3 - 27ux - 54v$ over $C : v^2 = u^3 + T$.

Degree 2 irreducibles in $\mathbb{F}_7[T]$

Irreducible $\mathfrak{p}$	$b_\mathfrak{p}$	Irreducible $\mathfrak{p}$	$b_\mathfrak{p}$	Irreducible $\mathfrak{p}$	$b_\mathfrak{p}$
T^2+1	98	T^2+2T+3	0	T^2+4T+6	0
T^2+2	98	T^2+2T+5	0	T^2+5T+2	-98
T^2+4	98	T^2+3T+1	-98	T^2+5T+3	0
T^2+T+3	0	T^2+3T+5	0	T^2+5T+5	0
T^2+T+4	-98	T^2+3T+6	0	T^2+6T+3	0
T^2+T+6	0	T^2+4T+1	-98	T^2+6T+4	-98
T^2+2T+2	-98	T^2+4T+5	0	T^2+6T+6	0

(Back to 8)