Localizations + Stable Weinstein Geometry

(Jt w/ O LAZAREV
Z SYLVAN)

[This talk is about interaction of the algebraic/categorical w/ geometry.]

<u>Localizations</u> (NOT in sense of QFT localizing to critical locus of a classical action, or sense of GW invariants localizing to S^1-fixed pts.)

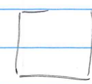
Given a ring R like $\mathbb{C}[x]$, and elements $S \subset R$, can create <u>universal</u> ring

$$R[S^{-1}]$$

in which S is multiplicatively inverted.

<u>Ex</u> $\mathbb{C}[x] = R$, $S = \{x^3 - 1\}$.
Then $R[S^{-1}]$ is $\mathbb{C}[x, \frac{1}{x^3-1}]$.

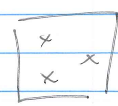
Generally, $\mathbb{C}[x] = \{\text{alg. fxns on } A^1\}$



$\mathbb{C} = A^1$

$\mathbb{C}[x][\frac{1}{x^3-1}] = \{\text{alg fxns on } A^1 \setminus \text{cube roots of } 1\}$

hence the term.



So "localization" is like restriction to the open subset $A^1 \setminus \{e^{2\pi n i/3}\}$.

Algebraically, we have, $\forall$ ~~other~~ rings T,

$$\{\text{ring homs } R[S^{-1}] \to T\} = \{\text{ring homs } R \to T \text{ for which } S \ni f \Rightarrow f \text{ is invertible in } T\}.$$

More generally, we can take any system w/ a good notion of identities (so we know what "inversion" is) and composition/multiplication, and ask for the universal object inverting a particular class of elements.

<u>Thm</u> (Dwyer-Kan, Quillen)

$$CW[eq^{-1}] \simeq Top$$

CW is collection (category) of spaces that can be given CW complex structures where morphisms are given by class of all continuous maps.

The ∞-category of spaces that can be given CW cplx structures. Same morphisms, but also encodes <u>topology</u> of spaces of maps (up to htpy equiv.)

Localize along (i.e. invert) those maps that are homotopy equivalences.

So localizing something <u>discrete</u> (LHS) can give interesting spaces (RHS).

<u>Cor</u> (by univ prop of localizations)
If T is a target category and $f: CW \to T$ is a functor carrying htpy equivalences to equivalences in T, then f extends to $\mathcal{T}op$.

(ex: T = chain cplxs, $f = C_*$ (singular chains).)

$$\begin{tikzcd} CW \arrow[r, "f"] \arrow[d] & T \\ \mathcal{T}op \arrow[ru, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

<u>Cor</u> Let G act continuously on X.
Then G acts continuously on $C_*(X)$.

Turns out such techniques also help in C^∞ <u>geometry</u>.
Our main result is about characterizing an ∞-category of interest as a localization.

<u>Thm</u> (Lazarev-Sylvan-T.)

$$\mathrm{Wein}^\diamond[\mathrm{eqs}^{-1}] \simeq \text{An } \infty\text{-category of (stabilized)}$$

Localization of a discrete category of (stabilized) Weinstein sectors.

Weinstein sectors w/ <u>spaces</u> of Weinstein embeddings.

I'll give some applications now, as though we're all experts. The rest of the talk will be explaining both sides of the equivalence.

<u>Cor</u> If $\mathrm{Wein}^\diamond \to T$ inverts equivalences of Weinstein sectors, it extends to a functor that's <u>continuous</u> on Weinstein embedding spaces.

An example is $\mathcal{W}$, the <u>wrapped</u> <u>Fukaya</u> <u>category</u> functor.

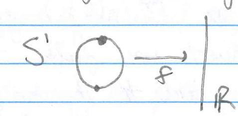
<u>Cor</u> If G acts (continuously) on a sector X by Weinstein automorphisms, then G acts (continuously) on $\mathcal{W}(X)$.

subtle to define; If G is connected, can in fact require instead G acts by <u>Liouville</u> automorphisms.

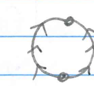
Before, writing a G-action on the wrapped category $\mathcal{W}(X)$ would require a lot of finagling with moduli of disks. We require none of that here.

There are other amazing things we can discover through localizations of $\mathrm{Wein}^\diamond$. Oleg might touch on this later this week.

Smooth, compact manifolds can be given handle decompositions (like cellular decompositions) using Morse theory. Here is a silly example:



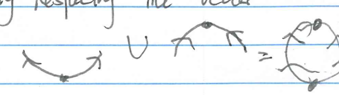
More fun; w/ a Riem metric, we have a ∇ vec field:



and near every crit pt of f, have a nbhd where ∇ looks standard:

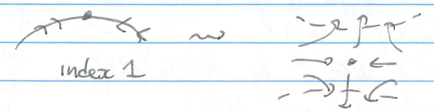
index 0 index 1.

We can glue nbhds to each other in a way respecting the vector fields to recover S^1 w/ Morse fcn:



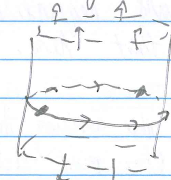
Apply T^* while adding p^2 to Morse fcn in local charts:

$h \cong q^2 + p^2$



$h \cong -q^2 + p^2$.

can still glue respecting vec fields, and we get



a model for T^*S!

Note p^2 gives conical structure near ∞.

Rmk The symp structure dpdq on T^*(nbhd) gives isom $T \cong T^*$, so can convert vec. fields to 1-forms. Can check the 1-form θ, dual to the vec fields, satisfies $d\theta = \omega$ and glues to a global 1-form (b/c vec fields are glued together).

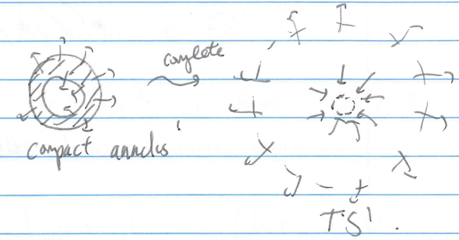
Weinstein mflds are those sympl. mflds (w/ data of vec field) arising in this way.

$\dim M = 2n$ 1-form

Defn A Weinstein manifold is a pair (M, θ) s.t.

(1) $d\theta = \omega$ is symplectic
(2) X_θ ($= \omega$-dual to θ) is ∇-like for some fcn h
(3) X_θ makes M a completion of a cmpct mfld w/ corners.

Ex T^*S^1 is completion of (cf (3))



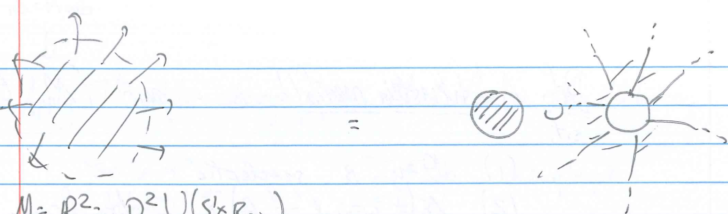
Defn A (strict) map of Weinstein mflds is a codim 0, C^∞, proper embedding $f: M \to N$ s.t. $f^*\theta^N = \theta^M$.

More genlly, a non-strict map allows for $f^*\theta^N = \theta^M + dh$ w/ h cmpctly supported (so conical structure near ∞ isn't affected).

Clearly, strict maps are quite rigid, so the space of strict maps isn't as rich. Most want to consider the space of non-strict maps.

A Weinstein sector is to Weinstein manifold as mflds-w/-corners are to manifolds.

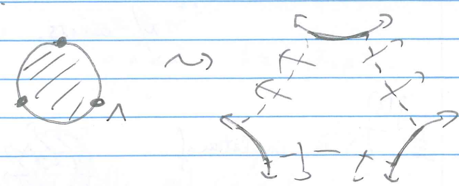
Constrn Given $M = M_{cmpct} \cup (\partial M_{cmpct}) \times \mathbb{R}_{\geq 0}$ using (3), and a Weinstein submfld $\Lambda \subset \partial(M_{cmpct})$, can make a mfld-w/ bdy w/ conical structure. (Next page)



$M = \mathbb{R}^2 = D^2 \cup (S^1 \times \mathbb{R}_{\geq 0})$

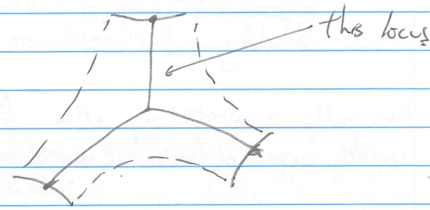
X_θ radially outward.

$\Lambda = 3$pts $\subset S^1$



new symp mfld, w/ ∂ given by $\Lambda \times \mathbb{R}$.

analogue of zero section of T^* is "locus where X_θ does not flow you to ∞":



Can think of Weinstein geometry as generalizing T^* to be T^* of "singular spaces" like the above "tripod."

Any symp-mfld-w/-θ made from a Weinstein mfld using above constxn iteratively is a Weinstein sector.

<u>Defn</u> An <u>equivalence</u> of Weinstein sectors is a (strict) map $f: M \to N$ s.t. $\exists$

- a not-nec.-strict map $g: N \to M$
- isotopies $fg \sim id_N$, $gf \sim id_M$ (though not nec. strict maps)
- $g^*\theta^M$ is Weinstein htpic to θ^N.

(i.e. $\exists \{\theta_t\}$ w/ $\theta_0 = \theta^N$, $\theta_1 = g^*\theta^M$, s.t. defines a Weinstein structure on $N \times T^*\Delta^1$).

Almost done explaining lefthand side of Thm. What is "$\circ$"?

<u>Rmk</u> For the wrapped Fukaya category $\mathcal{W}$, we know two things:

(1) $\mathcal{W}$ defines a functor

$$\mathrm{Wein} \longrightarrow A_\infty\mathrm{Cat}$$
$$M \longmapsto \mathcal{W}(M)$$

where each $f: M \to N$ induces $\mathcal{W}(M) \to \mathcal{W}(N)$

(2) If f is an equivalence, so is $\mathcal{W}(M) \to \mathcal{W}(N)$

(3). We have a natural equivalence $\mathcal{W}(M) \xrightarrow{\sim} \mathcal{W}(M \times T^*[0,1])$.

So, given (3), it's natural to define

$$\mathrm{Wein}^\circ := \bigcup \left(\mathrm{Wein} \xrightarrow{- \times T^*[0,1]} \mathrm{Wein} \xrightarrow{- \times T^*[0,1]} \cdots\right)$$

ie, $\mathcal{W}ein^{\diamond}$ is a category where M and $M \times T^*[0,1]$ are identified.

Rmk Note $- \times T^*[0,1]$ is NOT a functor if we allow for non-strict morphisms.

This explains the lefthand side, and we ran out of time together.

Let me say that (this is in these notes, but wasn't mentioned in the talks) the proof of the Thm proceeds in two steps

I. You prove the theorem for a category of all Liouville sectors (not just Weinstein), and strict maps, inverting all equivalences.

II. It turns out the Weinstein version (as in the Thm) sits inside the localization from I above, and miraculously the mapping spaces of II are nearly identical to those of I.

Thank you for listening/reading!

-HLT June 2022