Defects, branes and 3D lattice model

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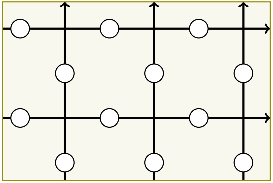
Lattice spin model in classical statistical mechanics:

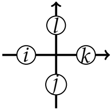


Spin variables ○ on edges interact at vertices ("vertex model").

A fundamental quantity is the partition function

$$Z = \sum_{\text{configs } s} e^{-E(s)/k_B T}$$

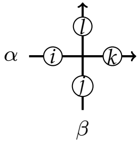
The Boltzmann weight is the product of local weights:



$$\rightsquigarrow \; R_{ij}^{kl}\,, \quad e^{-E(s)/k_B T} = \prod_{\text{vertices } v} R(v, s)$$

Rational 6-vertex model

- $\bigcirc \in \{+, -\}$ and only 6 configs allowed around each vertex
- Each line $\mathcal{L}_\alpha$ has a spectral parameter $z_\alpha \in \mathbb{C}$
- Local Boltzmann weights depend on the differences of z_α:



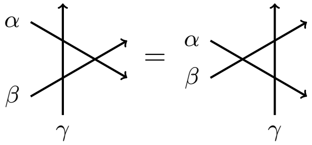
$$\rightsquigarrow \quad R(z_\alpha - z_\beta)_{ij}^{kl} \in \mathbb{C}$$

- The R-matrix $R(z) = (R(z)_{ij}^{kl}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is given by

$$R(z) = \frac{zI + \hbar P}{z + \hbar}, \quad P(v \otimes w) = w \otimes v$$

The 6-vertex R-matrix satisfies the Yang–Baxter equation

$$R_{\alpha\beta}(z_\alpha - z_\beta)R_{\alpha\gamma}(z_\beta - z_\gamma)R_{\beta\gamma}(z_\beta - z_\gamma)$$
$$= R_{\beta\gamma}(z_\beta - z_\gamma)R_{\alpha\gamma}(z_\beta - z_\gamma)R_{\alpha\beta}(z_\alpha - z_\beta)$$



YBE with spectral parameters implies integrability:

- 6-vertex model ↔ Heisenberg XXX quantum spin chain
- Commuting conserved charges act on the Hilbert space

6-vertex model appears in many supersymmetric QFTs:

- 2D $\mathcal{N} = (2,2)$ gauge theories [Nekrasov–Shatashvili]
- 4D $\mathcal{N} = 2$ gauge theories [Nekrasov–Shatashvili, Chen–Dorey–Hollowood–Lee]
- 3D $\mathcal{N} = 4$ gauge theories
  [Bullimore–Dimofte–Gaiotto, Braverman–Finkelberg–Nakajima, Dedushenko–Gaiotto]
- 4D $\mathcal{N} = 1$ gauge theories (with surface defects)
  [Gaiotto–Rastelli–Razamat, Gadde–Gukov, Gaiotto–Razamat, Maruyoshi–Y, Y]
- 4D $\mathcal{N} = 1$ gauge theories from brane tilings
  [Spiridonov, Bazhanov–Sergeev, Yamazaki, Y, ...]
- 4D Chern–Simons theory ($=$ Ω-deformed 6D MSYM)
  [Costello, Costello–Witten–Yamazaki]

All of these have brane construction in string theory.

They are related by string dualities. [Costello–Y]

Today: Lift this story to one dimension higher

- Brane setups in string theory → brane setup in M-theory
- 6-vertex model → integrable 3D lattice model

The two sides are related by dualities

The 3D model reduces to the 6-vertex model via circle reduction

Underlying principle:

topological quantum field theory

$+$

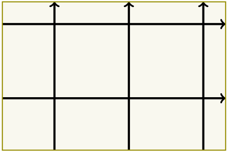
defects

$+$

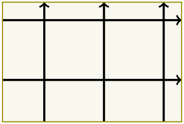
"extra dimensions"

Place a 2D TQFT with line defects on $\mathbb{T}^2$.

Wind line defects $\mathcal{L}_\alpha$, $\alpha = 1, \ldots, M + N$, around 1-cycles of $\mathbb{T}^2$:



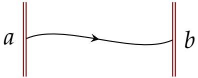
Task: calculate the correlator of the $M \times N$ lattice of line defects

$$\left\langle \prod_\alpha \mathcal{L}_\alpha \right\rangle = Z\left(\right.$$  $$\left. \right)$$

Strategy: chop up the torus into squares.

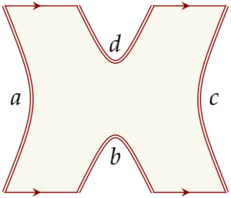
We use the formalism of open-closed TQFT.

Open string stretched between branes a and b:



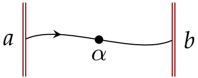
$\rightsquigarrow$ Hilbert space V_{ab}

Scattering of two open strings defines a map



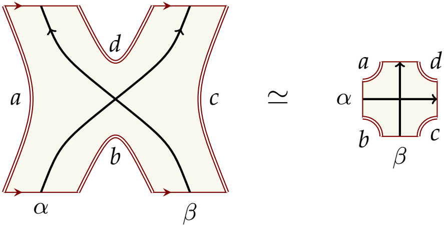
$: V_{ab} \otimes V_{bc} \to V_{ad} \otimes V_{dc}$

Open string with particle α attached, between branes a and b:



$\rightsquigarrow$ Hilbert space $V_{ab,\alpha}$

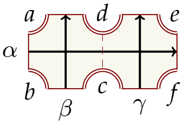
Scattering



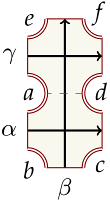
defines the R-matrix

$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix} : V_{ab,\alpha} \otimes V_{bc,\beta} \to V_{ad,\beta} \otimes V_{dc,\alpha}$$

Gluing pieces = composition of R-matrices:

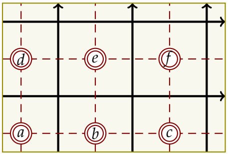


$$= \check{R}_{\alpha\gamma}\begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{dc,\alpha}} \check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$



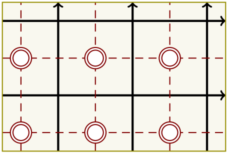
$$= \check{R}_{\alpha\gamma}\begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{ad,\beta}} \check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

Try to reconstruct the torus:



$$= \begin{matrix} & \mathrm{Tr} & \mathrm{Tr} & \cdots & \mathrm{Tr} \\ \mathrm{Tr} & \check{R} & \check{R} & \cdots & \check{R} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr} & \check{R} & \check{R} & \cdots & \check{R} \\ \mathrm{Tr} & \check{R} & \check{R} & \cdots & \check{R} \end{matrix}$$

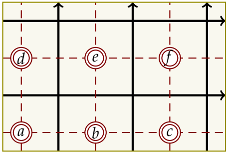
We got a slice of Swiss cheese!



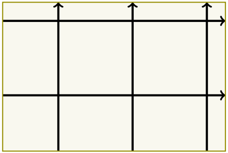
Assume: we have enough kinds of branes.

Then, summation over branes fill the holes:

$$\sum_{a,b,c,d,e,f}$$

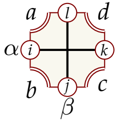


$$=$$

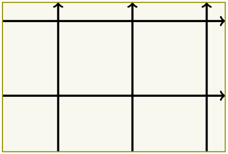


Represent matrix elements of $\check{R}$ by

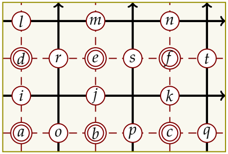
$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}^{lk}_{ij} =$$



We have found



$$= \sum_{a,b,c,d,e,f} \sum_{\substack{i,j,k,l,m,n \\ o,p,q,r,s,t}}$$



RHS is the partition function of a 2D lattice model!

$$\left\langle \prod \mathcal{L}_i \right\rangle = Z_{\text{2D lattice model}}$$

Spins ○ and ◎ interact at vertices with Boltzmann weights $\check{R}$.

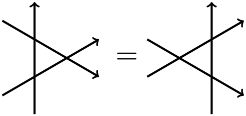
We can always absorb ◎ into ○; hereafter forget about ◎.

Now suppose there are hidden extra dimensions $\mathbb{M}$:

- Spacetime is really $\mathbb{T}^2 \times \mathbb{M}$
- The theory is topological on $\mathbb{T}^2$ but not on $\mathbb{M}$
- Line defects $\mathcal{L}_\alpha$ are located at points $u_\alpha \in \mathbb{M}$

Implications:

- $\mathcal{L}_\alpha = \mathcal{L}_\alpha(u_\alpha)$, $\check{R}_{\alpha\beta} = \check{R}_{\alpha\beta}(u_\alpha, u_\beta)$
- No singularities when lines are moved: YBE holds



Conclusion: TQFT + defects + extra dimensions imply the integrability of the lattice model

Example: 4D Chern–Simons theory

- Action:

$$S = \int_{\mathbb{T}^2 \times C} \mathrm{d}z \wedge \mathrm{CS}(A)\,, \quad A = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\bar{z}}\,\mathrm{d}\bar{z}$$

  where $C = \mathbb{C}$, cylinder or torus
- Topological on $\mathbb{T}^2$, holomorphic on C
- Wilson lines $\mathcal{L}_\alpha$ wrapping 1-cycles $\times \{z_\alpha\} \subset \mathbb{T}^2 \times C$

$\check{R}_{\alpha\beta}(z_\alpha, z_\beta) = \check{R}_{\alpha\beta}(z_\alpha - z_\beta)$ is a rational/trigonometric/elliptic solution of YBE.

For $G = \mathrm{SU}(2)$, $C = \mathbb{C}^\times$ and $\mathcal{L}_\alpha$ Wilson lines in the fundamental rep, we get the 6-vertex model.

The same argument holds in higher dimensions.

Setup:

- $(m+n)$-dim QFT on $\mathbb{T}^m \times \mathbb{M}_n$
- Topological on $\mathbb{T}^m$ but not on $\mathbb{M}_n$
- Defects $\mathcal{D}_\alpha$ making a lattice in $\mathbb{T}^m$, separated in $\mathbb{M}_n$

This structure implies

- Correlator is the partition function of a lattice model:

$$\left\langle \prod_\alpha \mathcal{D}_\alpha \right\rangle = Z_{\text{lattice model}}$$

- Integrability: m-dim analog of Yang–Baxter equation

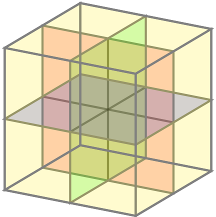
We will construct an integrable 3D lattice model using branes.

M5-brane system in M-theory:

	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L M5_1	—	$\cdot$	—	—	—	$\circ$	$\circ$	—	—	$\circ$	$\circ$
M M5_2	—	—	$\cdot$	—	$\circ$	—	$\circ$	—	—	$\circ$	$\circ$
N M5_3	—	—	—	$\cdot$	$\circ$	$\circ$	—	—	—	$\circ$	$\circ$

- — : M5 extends in that direction
- $\cdot$: M5 is located at any point
- $\circ$: M5 is located at the origin

M5s make an $L \times M \times N$ cubic lattice in $\mathbb{T}^3_{123}$



	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L M5$_1$	—	·	—	—	—	○	○	—	—	○	○
M M5$_2$	—	—	·	—	○	—	○	—	—	○	○
N M5$_3$	—	—	—	·	○	○	—	—	—	○	○

The supersymmetric index of the Hilbert space $\mathcal{H}$:

$$Z_{\text{M5}} := \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{i\theta(J_{78}-J_{9\natural})} e^{-\beta H}\right)$$

- $(-1)^F$: fermion parity
- H: Hamiltonian
- J_{78} & $J_{9\natural}$: rotation generators on $\mathbb{R}^2_{78}$ & $\mathbb{R}^2_{9\natural}$

By supersymmetry, only BPS states ($H = 0$) contribute:

$$Z_{\text{M5}} = \sum_{j=-\infty}^{\infty} q^j \, \mathrm{Tr}_{\mathcal{H}^j_{\text{BPS}}}(-1)^F =: \mathring{Z}_{\text{M5}} \,, \quad q = e^{i\theta} \,.$$

$\mathrm{Tr}_{\mathcal{H}^j_{\text{BPS}}}(-1)^F$ is the Witten index of $\mathcal{H}^j \subset \mathcal{H}$ in which $J_{78} - J_{9\natural} = j$.

	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L $\mathrm{M5}_1$	—	•	—	—	—	∘	∘	—	—	∘	∘
M $\mathrm{M5}_2$	—	—	•	—	∘	—	∘	—	—	∘	∘
N $\mathrm{M5}_3$	—	—	—	•	∘	∘	—	—	—	∘	∘

Generalization:

- Replace $\mathbb{T}^3_{123} \to M_3$ and $\mathbb{T}^3_{123} \times \mathbb{R}^3_{456} \to T^*M_3$
- Wrap M5s on conormals $N^*\Sigma_2 \subset T^*M_3$

$\mathring{Z}_{\mathrm{M5}}$ is still a q-series with the coefficients being Witten indices.

Witten indices are invariant under deformations of M_3 and M5s.

We get a structure of a TQFT on M_3 + surface defects.

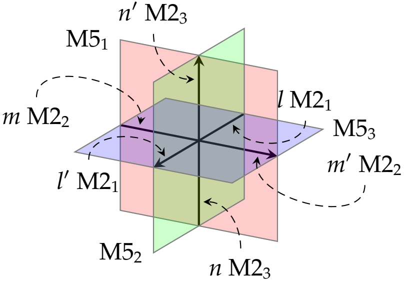
For $M_3 = \mathbb{T}^3$ and surface defects making a cubic lattice,

$$\mathring{Z}_{\text{M5}} = Z_{\text{3D lattice model}}$$

Spin variables: #M2-branes stretched between M5s

Local Boltzmann weights: 3D R-matrix $\mathcal{R}$

$$\mathcal{R}^{l'm'n'}_{lmn} =$$



Supersymmetric index:

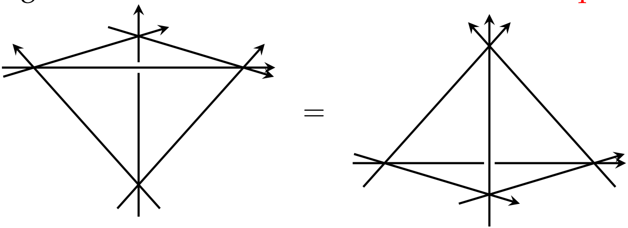
$$\mathring{Z}_{\text{M5}} = \sum_{\text{configs } s} \prod_{\text{vertices } v} \mathcal{R}(v, s)$$

	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
L M5_1	—	$\cdot$	—	—	—	$\cdot$	$\cdot$	—	—	$\circ$	$\circ$
M M5_2	—	—	$\cdot$	—	$\cdot$	—	$\cdot$	—	—	$\circ$	$\circ$
N M5_3	—	—	—	$\cdot$	$\cdot$	$\cdot$	—	—	—	$\circ$	$\circ$

M5s can be shifted in $\mathbb{R}^3_{456}$ and $\mathring{Z}_{\mathrm{M5}}$ still makes sense. Witten indices are invariant, hence so is $\mathring{Z}_{\mathrm{M5}}$.

$\mathbb{R}^3_{456}$ are extra dimensions!

3D analog of YBE is Zamolodchikov's tetrahedron equation:



Difference with 2D case: 3D R-matrix has no spectral parameter

This is because $\mathring{Z}_{\text{M5}}$ is independent of the M5 positions in $\mathbb{R}^3_{456}$.

But the supersymmetric index gets deformed:

$$Z_{\text{M5}} = \sum_j q^j \operatorname{Tr}_{\mathcal{H}^j_{\text{BPS}}} \left((-1)^F e^{-\beta (Z^{(2)}_{14} + Z^{(2)}_{25} + Z^{(2)}_{36})} \right),$$

where $Z^{(2)}$ is a 2-form charge.

It turns out

$$Z_{\text{M5}} = Z_{\text{3D lattice model with twisted B.C.}}$$

M5 positions in $\mathbb{R}^3_{456}$ determine twist of boundary conditions.

The brane construction implies a few key properties:

- Normalization: $R^{000}_{000} = 1$
- Parity reversal symmetry: $R^{l'm'n'}_{lmn} = R^{n'm'l'}_{nml}$
- Involutivity: $R = R^{-1}$
- Charge conservation: $R^{l'm'n'}_{lmn} = 0$ unless $l + m = l' + m'$ and $m + n = m' + n'$

The most remarkable is the behavior under reduction to 2D:

$$\mathcal{S}(z)^{\{l'_1,\ldots l'_N\}\{m'_1,\ldots m'_N\}}_{\{l_1,\ldots l_N\}\{m_1,\ldots m_N\}} = \sum_{n_1,\ldots,n_N} z^{n_1}\mathcal{R}^{l'_N m'_N n_1}_{l_N m_N n_N}\cdots\mathcal{R}^{l'_3 m'_3 n_3}_{l_2 m_2 n_2}\mathcal{R}^{l'_2 m'_2 n_2}_{l_1 m_1 n_1}$$

is the (trigonometric) 6-vertex R-matrix in the representation

$$\text{Fock space}^{\otimes N} = \bigoplus \text{symmetric tensor reps of } \mathrm{GL}(N)$$

This is deduced by relating the M5-brane system to a brane system for 4D CS by dualities.

A solution of TE that has all of these properties is known!

It was found by Kapranov–Voevodsky and Bazhanov–Sergeev:

$$\mathcal{R}_{lmn}^{l'm'n'} = \delta_{l+m}^{l'+m'} \delta_{m+n}^{m'+n'} \times \sum_{\substack{\lambda,\mu \in \mathbb{Z}_{\geq 0} \\ \lambda+\mu=m'}} (-1)^\lambda q^{l(n'-m)+(n+1)\lambda+\mu(\mu-n)} \frac{(q^2)_{n'+\mu}}{(q^2)_{n'}} \binom{l}{\mu}_{q^2} \binom{m}{\lambda}_{q^2}$$

where

$$(q)_n = \prod_{k=1}^{n} (1-q^k)\,, \quad \binom{m}{n}_q = \frac{(q)_m}{(q)_{m-n}(q)_n}$$

There is a $\mathrm{GL}(M|N)$ version of the story, which I conjecture to give R-matrices obtained by Sergeev and Yoneyama.

Summary

- TQFT + codim-1 defects produce lattice models.
- The existence of extra dimensions implies integrability.
- Integrable 3D lattice model can be constructed from branes.
- Related by dualities to 6-vertex models appearing in QFTs

Open problems:

- Quantitative determination of the 3D R-matrix?
- Other duality frames in which 3D lattice model appear?
- Beyond partition functions?