Torus knots, open Gromov-Witten invariants, and topological recursion

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Based on joint work with Bohan Fang

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Introduction of Gromov-Witten theory

GW theory: curve counting theory.

- Given $p_1 \neq p_2 \in \mathbb{R}^2$, there is a unique line $\ell \subset \mathbb{R}^2$ passing through p_1, p_2.
- Given $p_1 \neq p_2 \in \mathbb{P}^2$, there is a unique (complex projective) line $\ell \subset \mathbb{P}^2$ passing through p_1, p_2.
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- A general degree 2 homogeneous polynomials in X_0, X_1, X_2 is of the form $a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_0X_1 + a_4X_1X_2 + a_5X_0X_2$.
- The space of degree 2 nonzero homogeneous polynomials (modulo a global constant) can be identified with $\mathbb{P}^5 = \{[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]\}$.
- The space of smooth conics in $\mathbb{P}^2$ can be view as an open subset U in $\mathbb{P}^5$

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- The condition of passing through a given point corresponds to a hyperplane in $\mathbb{P}^5$. Since the 5 points are assumed to be in general position, the intersection of five such hyperplanes gives us a unique point.
- The points in $\mathbb{P}^5 \backslash U$ correspond to line pairs and double lines, and no such configuration can pass through 5 points, unless three of the points are collinear. $\Longrightarrow$ There is a unique smooth conic passing these 5 points.

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Introduction of Gromov-Witten theory

Using similar method, one can count plane cubics passing through 9 points, or more generally, plane curves of degree d passing through $d(d+3)/2$ points; in each case the answer is 1.

Another direction: Count degree d rational curves.

- Genus formula for nodal plane curves: $g = \frac{(d-1)(d-2)}{2} - \delta$, where δ is the number of nodes.
- Each node is a condition of codimension 1 and so we should consider the number of degree d rational curves passing through $d(d+3)/2 - \delta = 3d - 1$ points.

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Introduction of Gromov-Witten theory

Kontsevich's formula: Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane. Then the following recursive relation holds:

$$N_d + \sum_{d_1+d_2=d, d_1, d_2 \geqslant 1} \frac{(3d-4)!}{(3d_1-1)!(3d-3d_1-3)!} d_1^3 N_{d_1} N_{d_2} d_2$$

$$= \sum_{d_1+d_2=d, d_1, d_2 \geqslant 1} \frac{(3d-4)!}{(3d_1-2)!(3d-3d_1-2)!} d_1^2 N_{d_1} d_2^2 N_{d_2}$$

Initial condition: $N_1 = 1$.

Method: Use Gromov-Witten invariants: Count maps $f: C \to X$ from algebraic curve C to a certain target space X.

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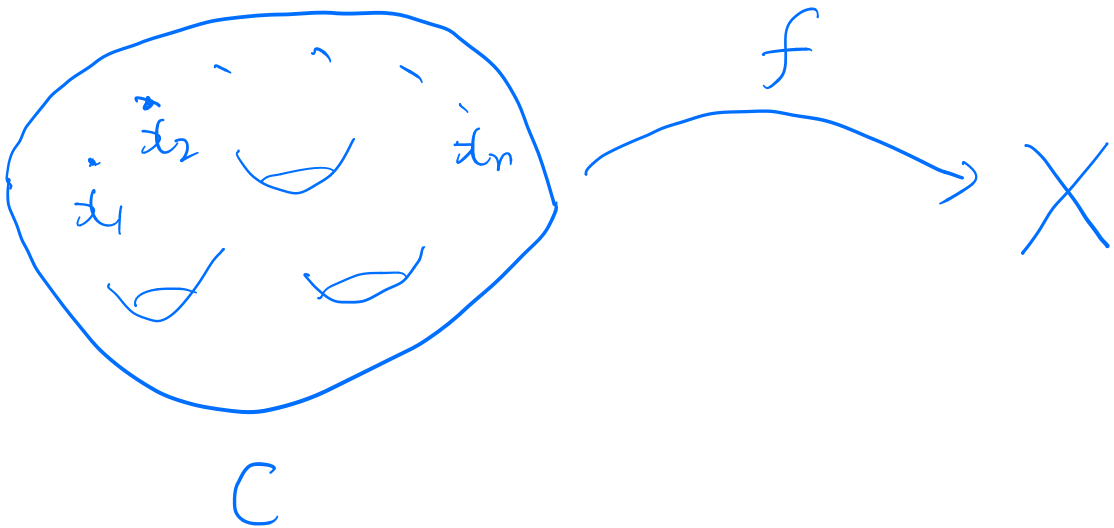

Introduction of Gromov-Witten theory

Let $\mathcal{X}$ be a symplectic manifold and let $\mathcal{L} \subset \mathcal{X}$ be a Lagrangian sub-manifold.

Sometimes we are also interested in Open Gromov-Witten invariants: Count maps $f : C \to \mathcal{X}$, where C is a genus g bordered Riemann surface with n boundary circles such that f(boundary circles)$\subset \mathcal{L}$.

Introduction of Gromov-Witten theory

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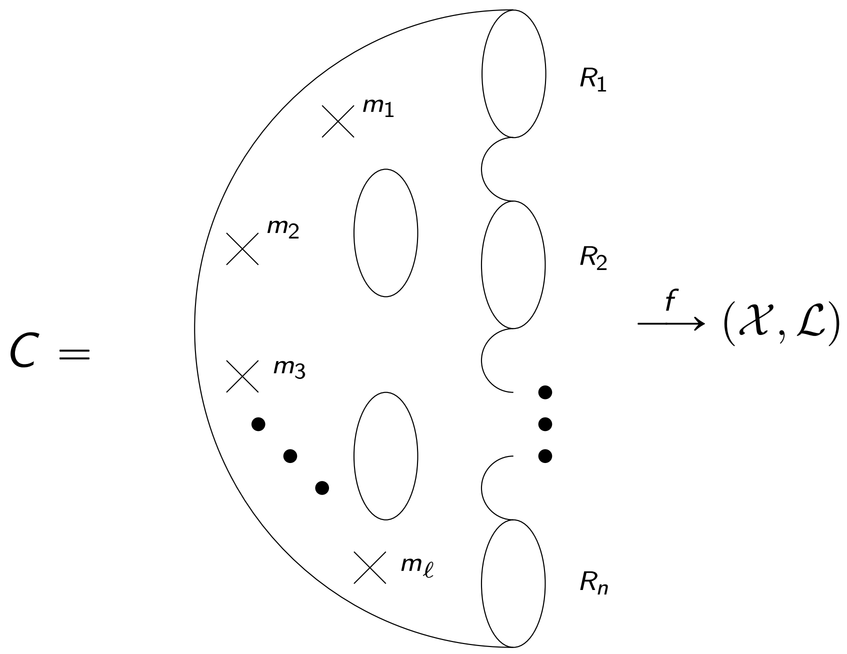
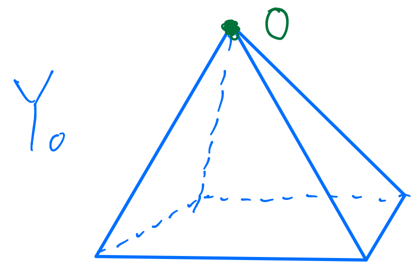


Conifold transition



Consider the **conifold** $\mathcal{Y}_0$ defined as

$$\mathcal{Y}_0 := \{(x, y, z, w) \in \mathbb{C}^4 \mid xz - yw = 0\}. \quad (1)$$

it has a unique singularity at the origin.

Two ways to smooth the singularity:

- To deform the singularity $\rightsquigarrow$ **deformed conifold**
- To resolve the singularity $\rightsquigarrow$ **resolved conifold**

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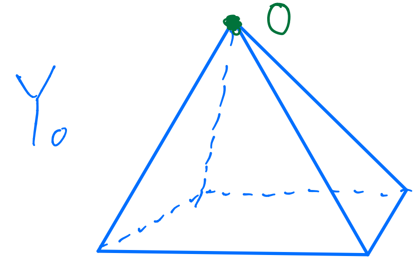

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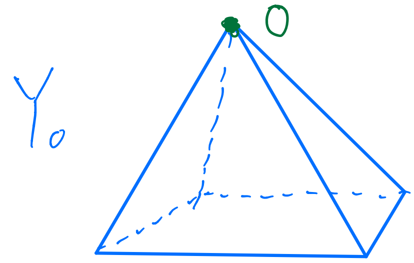

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Let δ be a small positive number. Consider the deformed conifold $\mathcal{Y}_\delta$ defined as

$$\mathcal{Y}_\delta := \{(x, y, z, w) \in \mathbb{C}^4 \mid xz - yw = \delta\}. \tag{2}$$

$\Longrightarrow \mathcal{Y}_\delta$ is smooth.

Consider the standard symplectic form on $\mathbb{C}^4$:

$$\omega_{\mathbb{C}^4} = \frac{\sqrt{-1}}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z} + dw \wedge d\bar{w}).$$

The symplectic form on $\mathcal{Y}_\delta$ is defined as $\omega_{Y_\delta} := \omega_{\mathbb{C}^4}|_{Y_\delta}$.

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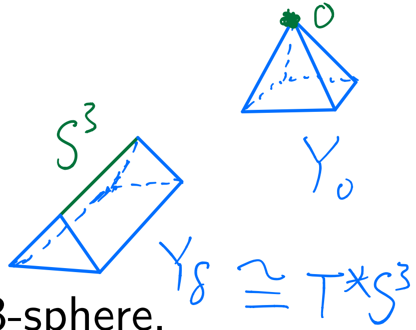
Conifold transition

There exists a symplectomorphism

$$\phi_\delta : \mathcal{Y}_\delta \to T^*S^3,$$



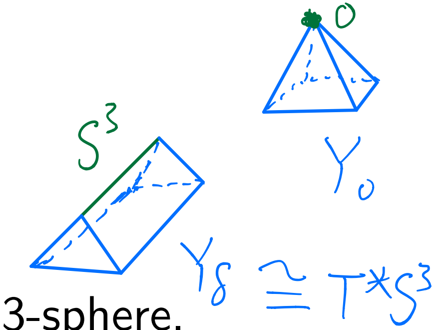
where T^*S^3 is the cotangent bundle of the 3-sphere.

Consider the anti-holomorphic involution

$$\begin{aligned} I : \mathbb{C}^4 &\to \mathbb{C}^4 \\ (x, y, z, w) &\mapsto (\bar{z}, -\bar{w}, \bar{x}, -\bar{y}). \end{aligned} \tag{3}$$

Then $\mathcal{Y}_\delta$ is preserved by I. The fixed locus S_δ of the induced anti-holomorphic involution I_δ on $\mathcal{Y}_\delta$ is a 3-sphere of radius $\sqrt{\delta}$ and $\phi_\delta(S_\delta)$ is the zero section of T^*S^3. When $\delta \to 0$, S_δ shrinks to the unique singular point of $\mathcal{Y}_0$.

Conifold transition



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Conifold transition

The second way to smooth the singularity of $\mathcal{Y}_0$ is to consider the **resolved conifold** $\mathcal{X}$. We consider the blow-up of $\mathbb{C}^4$ along the subspace $\{(x, y, z, w)|y = z = 0\}$. Let $\mathcal{X}$ be the resolution of $\mathcal{Y}_0$ under the blow-up. Then $\mathcal{X}$ is isomorphic to the local $\mathbb{P}^1$:

$$\mathcal{X} \cong \mathrm{Tot}[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1].$$

If we view $\mathcal{X}$ as a subspace of $\mathbb{C}^4 \times \mathbb{P}^1$, then $\mathcal{X}$ is defined by the following equations:

$$xs = wt, \quad ys = zt,$$

where $[s : t]$ is the homogeneous coordinate on $\mathbb{P}^1$. The resolution $p : \mathcal{X} \to \mathcal{Y}_0$ is given by contracting the base $\mathbb{P}^1$ in $\mathcal{X}$. We say that $\mathcal{X}$ and $\mathcal{Y}_\delta$ are related by the **conifold transition**.

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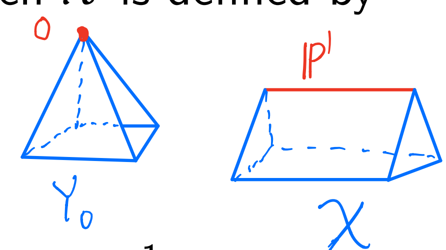

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Torus knots

- A knot $K \subset S^3$: an isotopy class of embeddings of S^1 in S^3.
- Let $P, Q \in \mathbb{Z}_{>0}$ with $\gcd(P, Q) = 1$. Let

$$\begin{aligned} K : S^1 &\to S^1 \times S^1 \subset \mathbb{R}^3 \subset S^3 \\ z &\mapsto (z^P, z^Q). \end{aligned}$$

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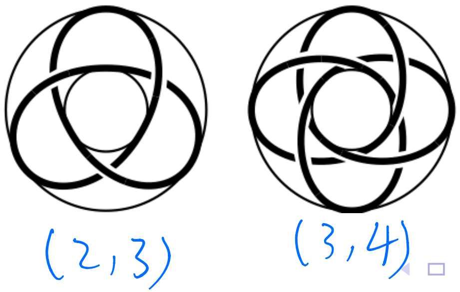

Torus knots

- Consider the conormal bundle N_K^* of $K \subset S^3$ defined as

$$N_K^* = \{(u, v) \in T^*S^3 : u = K(t), \quad \langle v, K'(t) \rangle = 0\},$$

  where $K'(t)$ is the derivative of K and $\langle , \rangle$ is the natural pairing between tangent and cotangent vectors. Then N_K^* is a Lagrangian sub-manifold of T^*S^3.
- We want to obtain a Lagrangian sub-manifold in the resolved conifold $\mathcal{X}$ from N_K^* under the conifold transition.
- Difficulty: the intersection of N_K^* with the zero section is non-empty.
- Solution: Diaconescu-Shende-Vafa 11 $\rightsquigarrow$ we can fiberwisely translate N_K^* to obtain a new Lagrangian sub-manifold M_K such that M_K does not intersect with the zero section.

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$$xs = wt, \quad ys = zt.$$

For $\epsilon > 0$, we consider the symplectic form $(\omega_{\mathbb{C}^4} + \epsilon^2 \omega_{\mathbb{P}^1})$ on $\mathbb{C}^4 \times \mathbb{P}^1$. Define the symplectic form $\omega_{\mathcal{X},\epsilon}$ on $\mathcal{X}$ by

$$\omega_{\mathcal{X},\epsilon} := (\omega_{\mathbb{C}^4} + \epsilon^2 \omega_{\mathbb{P}^1})\,|_{\mathcal{X}}\ .$$

Torus knots

Let $B(\epsilon)=\{(y,z)\in\mathbb{C}^2 \mid |y|^2+|z|^2\leqslant\epsilon^2\}\subset\mathbb{C}^2$ be the ball of radius ϵ. Consider the radial map $\rho_\epsilon:\mathbb{C}^2\backslash\{0\}\to\mathbb{C}^2\backslash B(\epsilon)$,

$$\rho_\epsilon(y,z)=\frac{\sqrt{|y|^2+|z|^2+\epsilon^2}}{\sqrt{|y|^2+|z|^2}}(y,z).$$

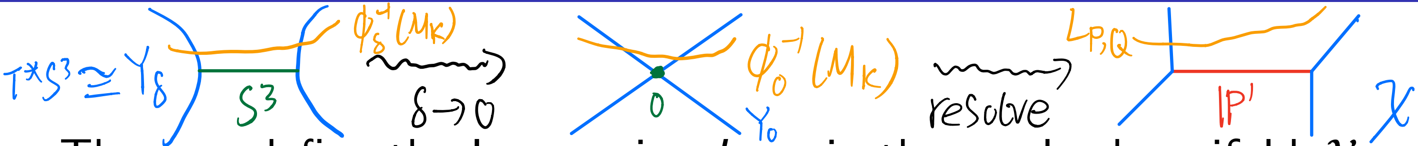
Let $\varrho_\epsilon=\mathrm{id}_{\mathbb{C}^2}\times\rho_\epsilon:\mathbb{C}^2\times(\mathbb{C}^2\backslash\{0\})\to\mathbb{C}^2\times(\mathbb{C}^2\backslash B(\epsilon))$. Then ϱ_ϵ preserves the conifold $\mathcal{Y}_0$ and it maps $\mathcal{Y}_0\backslash\{0\}$ to $\mathcal{Y}_0(\epsilon):=\mathcal{Y}_0\backslash(\mathcal{Y}_0\cap(\mathbb{C}^2\times B(\epsilon)))$.

McDuff-Salamon 98 $\Longrightarrow$ the map

$$\psi_\epsilon:=\varrho_\epsilon|_{\mathcal{Y}_0\backslash\{0\}}\circ p|_{\mathcal{X}\backslash\mathbb{P}^1}:\mathcal{X}\backslash\mathbb{P}^1\to\mathcal{Y}_0(\epsilon)$$

is a symplectomorphism.

Torus knots



Then we define the Lagrangian $L_{P,Q}$ in the resolved conifold $\mathcal{X}$ to be

$$L_{P,Q} := \psi_\epsilon^{-1}(\phi_0^{-1}(M_K)).$$

Recall that $\phi_0 : \mathcal{Y}_0 \backslash \{0\} \to T^*S^3 \backslash S^3$ is a symplectomorphism.

A nice property: consider the S^1-action on $\mathcal{X}$ defined as

$$u \cdot ((x, y, z, w), [s : t]) = ((u^Q x, u^P y, u^{-Q} z, u^{-P} w), [u^{-P-Q} s : t]).$$

Then $L_{P,Q}$ is preserved by the above action. $\rightsquigarrow$ We can use virtual localization techniques to study the open Gromov-Witten theory of $(\mathcal{X}, L_{P,Q})$.

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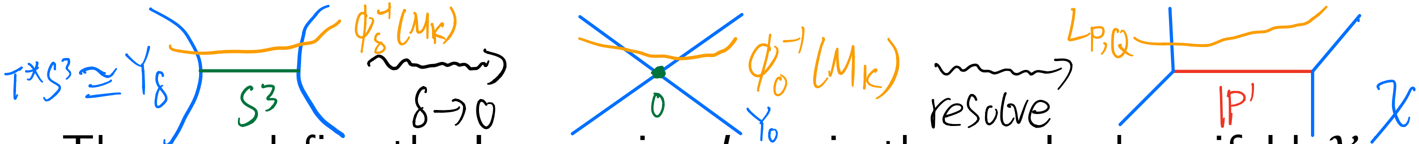

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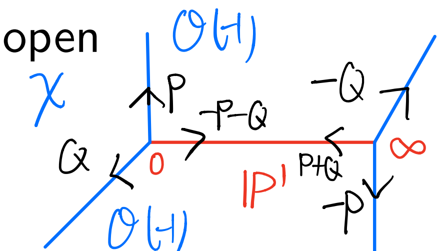

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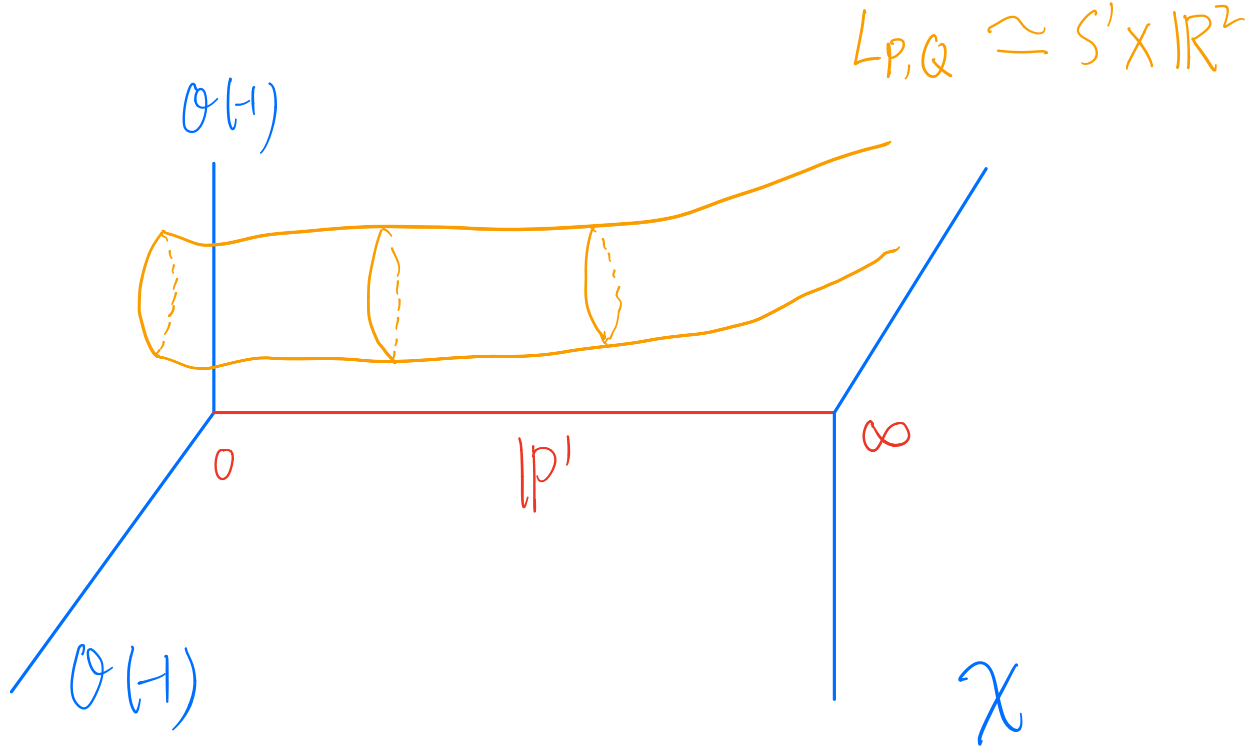
$L_{P,Q} \simeq S^1 \times \mathbb{R}^2$
$\mathcal{O}(-1)$
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0
$\mathbb{P}^1$
∞
X

A-model

- $\mathcal{X}$ The resolved conifold.
- $L_{P,Q} \subset \mathcal{X}$ the Lagrangian sub-manifold constructed from the torus knot.
- Consider the open Gromov-Witten potential $F_{g,n}^{(\mathcal{X},L_{P,Q})}$ of $(\mathcal{X}, L_{P,Q})$.

Open Gromov-Witten invariants: Count maps $f : C \to \mathcal{X}$, where C is a genus g bordered Riemann surface with n boundary circles such that f(boundary circles)$\subset L_{P,Q}$.

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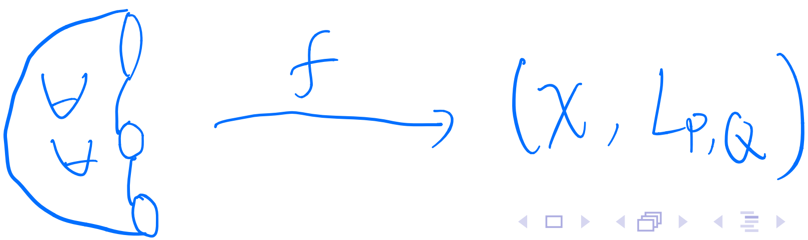

B-model

- Q: Higher genus B-model?
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- Let $C \subset (\mathbb{C}^*)^2$ be a curve and let X, Y be two meromorphic functions on C.
- Critical (ramification) points P_α of X: $dX = 0$.
- $X = e^{-x}, \quad Y = e^{-y}$.
- Near each ramification point P_α, use local coordinates:

$$x = x_0 + \zeta_\alpha^2, \quad y = y_0 + \sum_{i=1}^{\infty} h_i^\alpha \zeta_\alpha^i.$$

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The initial data of the topological recursion

The initial data of the topological recursion is given by $\omega_{0,1}, \omega_{0,2}$.

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- $A_i \cap B_j = \delta_{ij}$, $A_i \cap A_j = 0$, $B_i \cap B_j = 0$.

Fundamental differential of the second kind (a.k.a. Bergmann kernel) $\omega_{0,2}(p_1, p_2)$: symmetric 2-form on $\bar{C} \times \bar{C}$. It is uniquely characterized by

- $$\int_{p_1 \in A_i} \omega_{0,2}(p_1, p_2) = 0;$$
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Chekhov-Eynard-Orantin construct symmetric forms $\omega_{g,n}$ on C^n:

- Initial data $\omega_{0,1} = ydx$, $\omega_{0,2}$ as above;
- The recursive algorithm is:

$$\begin{aligned}
&\omega_{g,n+1}(p_0,\dots,p_n) \\
&= \sum_{P_\alpha} \mathrm{Res}_{p\to P_\alpha} \frac{\int_{\bar p}^{p} \omega_{0,2}(p_0,\cdot)}{2(y(p)-y(\bar p))dx(p)} \\
&\cdot \Big(\omega_{g-1,n+2}(p,\bar p,p_1,\dots,p_n) \\
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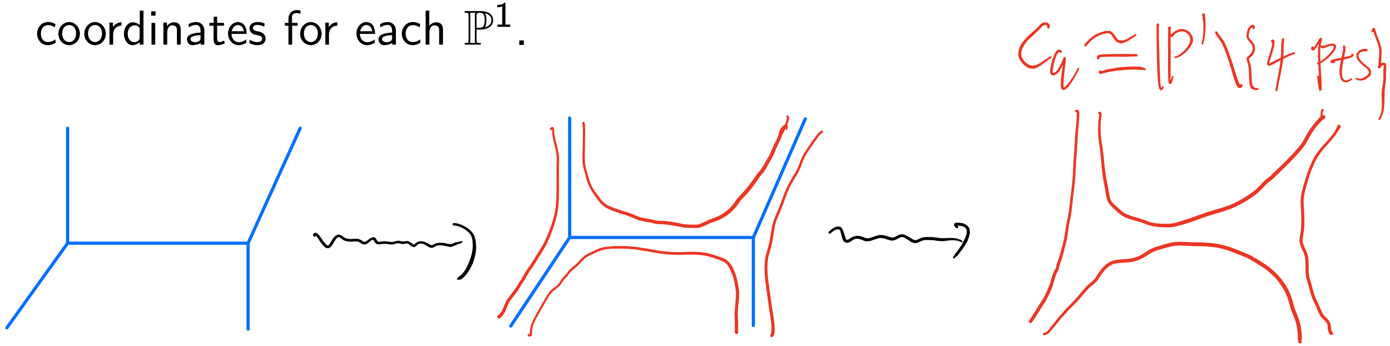
Mirror curve

The mirror curve $C_q \subset (\mathbb{C}^*)^2$ of $\mathcal{X}$ is defined by the following equation

$$1 + U + V + qUV = 0$$

in $(\mathbb{C}^*)^2$. Here q is a parameter on B-model. This curve allows a compactification into a genus 0 projective curve $\overline{C}_q$ in $\mathbb{P}^1 \times \mathbb{P}^1$, where $(1 : U)$ and $(1 : V)$ are homogeneous coordinates for each $\mathbb{P}^1$.



Mirror curve

Since $\gcd(P, Q) = 1$, we choose $\gamma, \delta \in \mathbb{Z}$ (not uniquely) chosen such that

$$\begin{pmatrix} Q & P \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}).$$

Consider the following change of variables

$$X = U^Q V^P, \quad Y = U^\gamma V^\delta.$$

We define the spectral curve as the quadruple

$$(C_q \subset (\mathbb{C}^*)^2, \overline{C}_q \subset \mathbb{P}^1 \times \mathbb{P}^1, X, Y).$$

The variables X, Y are holomorphic functions on C_q and meromorphic on $\overline{C}_q$.

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- Initial data $\omega_{0,1} = ydx$;
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$$\omega_{0,2} = \frac{dU_1 dU_2}{(U_1 - U_2)^2}.$$

The choice of $\omega_{0,2}$ involves a symplectic basis on $H_1(\overline{C}_q;\mathbb{C})$ – since the genus of our $\overline{C}_q$ is 0, this extra piece of datum is not needed.

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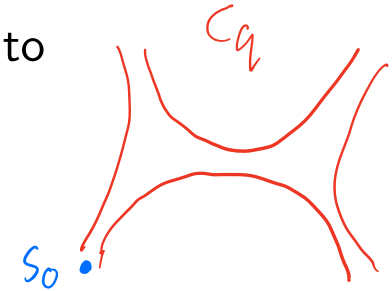

$$X = -V^P \left(\frac{V+1}{1+qV} \right)^Q .$$

Let $\eta = X^{\frac{1}{Q}}$. Then η is a local coordinate for the mirror curve $\overline{C}_q$ around $\mathfrak{s}_0 = (X, V) = (0, -1)$.

There exists $\delta > 0$ and $\epsilon > 0$ such that for $|q| < \epsilon$, the function η is well-defined and restricts to an isomorphism

$$\eta : D_q \to D_\delta = \{\eta \in \mathbb{C} : |\eta| < \delta\},$$

where $D_q \subset \overline{C}_q$ is an open neighborhood of $\mathfrak{s}_0$. Denote the inverse map of η by ρ_q and

$$\rho_q^{\times n} = \rho_q \times \cdots \times \rho_q : (D_\delta)^n \to (D_q)^n \subset (\overline{C}_q)^n.$$

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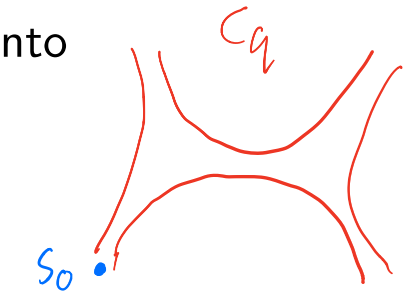

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Mirror curve

We define

$$W_{g,n}(\eta_1, \ldots, \eta_n, q) = \int_0^{\eta_1} \cdots \int_0^{\eta_n} (\rho_q^{\times n})^* \omega_{g,n}, \quad 2g - 2 + n > 0.$$

Let $f \in \mathbb{C}[[\eta_1, \ldots, \eta_n]]$. For a fixed integer number $Q > 0$, we denote

$$\mathfrak{h} \cdot f(\eta_1, \ldots, \eta_n) = \sum_{k_1, \ldots, k_n = 0}^{Q-1} \frac{f(a^{k_1}\eta_1, \ldots, a^{k_n}\eta_n)}{Q^n},$$

where a is a primitive Q-th root of unity. This operation "throws away" all terms with degree not divisible by Q.

Mirror symmetry

Theorem

Under the mirror map

$$F_{g,n}^{\mathcal{X},L_{P,Q}} = (-1)^{g-1+n} Q^n (\mathfrak{h} \cdot W_{g,n})(\eta_1, \ldots, \eta_n, q).$$

In other words, $F_{g,n}^{\mathcal{X},L_{P,Q}}$ *is equal to the part in the power series expansion of* $(-1)^{g-1+n} Q^n W_{g,n}(q, \eta_1, \ldots, \eta_n)$ *whose degrees of each* η_k *are divisible by* Q*.*

Some remarks

Remark

The above theorem is an all genus open-closed mirror symmetry between the Gromov-Witten theory of $(\mathcal{X}, L_K)$ *and the topological recursion of the mirror curve.*

On the other hand, Borot-Eynard-Orantin 13 $\Longrightarrow$ *Topological recursion is equivalent to the colored HOMFLY polynomial of the knot* K.

Therefore, the following three objects are equivalent:

1. *the open-closed Gromov-Witten invariants of* (X, L_K);
2. *the Eynard-Orantin invariants of the mirror curve;*
3. *the colored HOMFLY polynomial of the knot* K.

The equivalence of colored HOMFLY and GW is called large N duality.

Some remarks

When $P = Q = 1$, the Lagrangian $L_{1,1}$ is called an **Aganagic-Vafa brane**. The large N duality for $(\mathcal{X}, L_{1,1})$ is also called the **Mariño-Vafa formula**. An Aganagic-Vafa brane $\mathcal{L}$ can be defined for any toric Calabi-Yau 3-folds/3-orbifolds $\widetilde{\mathcal{X}}$.

Bouchard-Klemm-Marino-Pasquetti 07, 08: Introduce the Remodeling Conjecture

Theorem (Remodeling Conjecture)

If we expand $\omega_{g,n}$ under suitable local coordinate on the mirror curve C of $\widetilde{\mathcal{X}}$, we obtain the open Gromov-Witten potential $F_{g,n}^{(\widetilde{\mathcal{X}},\mathcal{L})}$ under the mirror map.

Remodeling Conjecture

- When $\tilde{\mathcal{X}} = \mathbb{C}^3$, open part: L. Chen, J. Zhou; closed part: Bouchard-Catuneanu-Marchal-Sułkowski, S. Zhu.
- When $\tilde{\mathcal{X}}$ is smooth: Eynard-Orantin.
- General semi-projective toric CY 3-orbifolds: Fang-Liu-Z.

If we start from $(\mathcal{X}, L_{1,1})$, then the Remodeling conjecture and the topic today can be viewed as generalizations along two different directions:

- Remodeling conjecture: generalizes the ambient space: resolved conifold $\rightsquigarrow$ toric CY 3-orbifolds.
- Today's topic: generalizes the Lagrangian sub-manifold: $L_{1,1} \rightsquigarrow L_{P,Q}$ (trivial knot $\rightsquigarrow$ torus knots).

Thanks

Thank you!