Torus knots, open Gromov-Witten invariants, and topological recursion

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Based on joint work with Bohan Fang
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Introduction of Gromov-Witten theory

**GW theory**: curve counting theory.

- Given $p_1 \neq p_2 \in \mathbb{R}^2$, there is a unique line $\ell \subset \mathbb{R}^2$ passing through $p_1, p_2$.
- Given $p_1 \neq p_2 \in \mathbb{P}^2$, there is a unique (complex projective) line $\ell \subset \mathbb{P}^2$ passing through $p_1, p_2$.
- Given 5 points in general position (any 3 points are not collinear) in $\mathbb{P}^2$, how many smooth conics pass through these 5 points?
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**Introduction of Gromov-Witten theory**

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A general degree 2 homogeneous polynomials in $X_0, X_1, X_2$ is of the form

$$a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_0X_1 + a_4X_1X_2 + a_5X_0X_2.$$ 

The space of degree 2 nonzero homogeneous polynomials (modulo a global constant) can be identified with

$$\mathbb{P}^5 = \{[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]\}.$$ 

The space of smooth conics in $\mathbb{P}^2$ can be view as an open subset $U$ in $\mathbb{P}^5$. 

Introduction of Gromov-Witten theory
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Introduction of Gromov-Witten theory

- The condition of passing through a given point corresponds to a hyperplane in $\mathbb{P}^5$. Since the 5 points are assumed to be in general position, the intersection of five such hyperplanes gives us a unique point.

- The points in $\mathbb{P}^5 \setminus U$ correspond to line pairs and double lines, and no such configuration can pass through 5 points, unless three of the points are collinear. $\implies$ There is a unique smooth conic passing these 5 points.
Introduction of Gromov-Witten theory

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Using similar method, one can count plane cubics passing through 9 points, or more generally, plane curves of degree $d$ passing through $d(d + 3)/2$ points; in each case the answer is 1.

Another direction: Count degree $d$ rational curves.

- Genus formula for nodal plane curves: $g = \frac{(d-1)(d-2)}{2} - \delta$, where $\delta$ is the number of nodes.
- Each node is a condition of codimension 1 and so we should consider the number of degree $d$ rational curves passing through $d(d + 3)/2 - \delta = 3d - 1$ points.
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Kontsevich’s formula: Let $N_d$ be the number of rational curves of degree $d$ passing through $3d - 1$ general points in the plane. Then the following recursive relation holds:

$$N_d + \sum_{d_1 + d_2 = d, d_1, d_2 \geq 1} \frac{(3d - 4)!}{(3d_1 - 1)!(3d - 3d_1 - 3)!} d_1^3 N_{d_1} N_{d_2} d_2 = \sum_{d_1 + d_2 = d, d_1, d_2 \geq 1} \frac{(3d - 4)!}{(3d_1 - 2)!(3d - 3d_1 - 2)!} d_1^2 N_{d_1} d_2^2 N_{d_2}$$

Initial condition: $N_1 = 1$.

Method: Use Gromov-Witten invariants: Count maps $f : C \rightarrow X$ from algebraic curve $C$ to a certain target space $X$. 
**Introduction of Gromov-Witten theory**

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**Torus knots**

**Introduction**

**Torus knots and conifold transition**

**Open Gromov-Witten theory and topological recursion**
Introduction of Gromov-Witten theory
Let $\mathcal{X}$ be a symplectic manifold and let $\mathcal{L} \subset \mathcal{X}$ be a Lagrangian sub-manifold.

Sometimes we are also interested in **Open Gromov-Witten invariants**: Count maps $f : C \to \mathcal{X}$, where $C$ is a genus $g$ **bordered** Riemann surface with $n$ boundary circles such that $f(\text{boundary circles}) \subset \mathcal{L}$.
Introduction of Gromov-Witten theory
Introduction of Gromov-Witten theory
Consider the **conifold** $\mathcal{Y}_0$ defined as

$$\mathcal{Y}_0 := \{(x, y, z, w) \in \mathbb{C}^4 \mid xz - yw = 0\}.$$  

(1)

it has a unique singularity at the origin.

Two ways to smooth the singularity:

- To deform the singularity $\rightsquigarrow$ **deformed conifold**
- To resolve the singularity $\rightsquigarrow$ **resolved conifold**
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- To deform the singularity $\rightsquigarrow$ **deformed conifold**
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Let $\delta$ be a small positive number. Consider the deformed conifold $\mathcal{Y}_\delta$ defined as

$$
\mathcal{Y}_\delta := \{(x, y, z, w) \in \mathbb{C}^4 \mid xz - yw = \delta\}. 
$$

$\implies \mathcal{Y}_\delta$ is smooth.

Consider the standard symplectic form on $\mathbb{C}^4$:

$$
\omega_{\mathbb{C}^4} = \frac{\sqrt{-1}}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z} + dw \wedge d\bar{w}).
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The symplectic form on $\mathcal{Y}_\delta$ is defined as $\omega_{\mathcal{Y}_\delta} := \omega_{\mathbb{C}^4} \mid \mathcal{Y}_\delta$. 

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The symplectic form on $Y_\delta$ is defined as $\omega_{Y_\delta} := \omega_{\mathbb{C}^4} \mid_{Y_\delta}$. 

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**Conifold transition**

Torus knots

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Conifold transition

There exists a symplectomorphism

$$\phi_\delta : \mathcal{Y}_\delta \to T^*S^3,$$

where $T^*S^3$ is the cotangent bundle of the 3-sphere.

Consider the anti-holomorphic involution

$$I : \mathbb{C}^4 \to \mathbb{C}^4$$

$$(x, y, z, w) \mapsto (\bar{z}, -\bar{w}, \bar{x}, -\bar{y}).$$

Then $\mathcal{Y}_\delta$ is preserved by $I$. The fixed locus $S_\delta$ of the induced anti-holomorphic involution $I_\delta$ on $\mathcal{Y}_\delta$ is a 3-sphere of radius $\sqrt{\delta}$ and $\phi_\delta(S_\delta)$ is the zero section of $T^*S^3$. When $\delta \to 0$, $S_\delta$ shrinks to the unique singular point of $\mathcal{Y}_0$. 
Conifold transition

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Conifold transition

The second way to smooth the singularity of $\mathcal{Y}_0$ is to consider the resolved conifold $\mathcal{X}$. We consider the blow-up of $\mathbb{C}^4$ along the subspace $\{(x, y, z, w) | y = z = 0\}$. Let $\mathcal{X}$ be the resolution of $\mathcal{Y}_0$ under the blow-up. Then $\mathcal{X}$ is isomorphic to the local $\mathbb{P}^1$: 

$$\mathcal{X} \cong \text{Tot}[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1].$$

If we view $\mathcal{X}$ as a subspace of $\mathbb{C}^4 \times \mathbb{P}^1$, then $\mathcal{X}$ is defined by the following equations:

$$xs = wt, \quad ys = zt,$$

where $[s : t]$ is the homogeneous coordinate on $\mathbb{P}^1$. The resolution $p : \mathcal{X} \to \mathcal{Y}_0$ is given by contracting the base $\mathbb{P}^1$ in $\mathcal{X}$. We say that $\mathcal{X}$ and $\mathcal{Y}_\delta$ are related by the conifold transition.
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A knot \( K \subset S^3 \): an isotopy class of embeddings of \( S^1 \) in \( S^3 \).

Let \( P, Q \in \mathbb{Z}_{>0} \) with \( \gcd(P, Q) = 1 \). Let

\[
K : S^1 \rightarrow S^1 \times S^1 \subset \mathbb{R}^3 \subset S^3
\]

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z \mapsto (z^P, z^Q).
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Then \( K \) is called a \((P, Q)\)-torus knot.
Torus knots

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Torus knots

Consider the conormal bundle \( N_K^* \) of \( K \subset S^3 \) defined as

\[
N_K^* = \{(u, v) \in T^*S^3 : u = K(t), \quad \langle v, K'(t) \rangle = 0 \},
\]

where \( K'(t) \) is the derivative of \( K \) and \( \langle , \rangle \) is the natural pairing between tangent and cotangent vectors. Then \( N_K^* \) is a Lagrangian sub-manifold of \( T^*S^3 \).

We want to obtain a Lagrangian sub-manifold in the resolved conifold \( X \) from \( N_K^* \) under the conifold transition.

Difficulty: the intersection of \( N_K^* \) with the zero section is non-empty.

Solution: Diaconescu-Shende-Vafa we can fiberwisely translate \( N_K^* \) to obtain a new Lagrangian sub-manifold \( M_K \) such that \( M_K \) does not intersect with the zero section.
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$$xs = wt, \quad ys = zt.$$ 

For $\epsilon > 0$, we consider the symplectic form $(\omega_{\mathbb{C}^4} + \epsilon^2 \omega_{\mathbb{P}^1})$ on $\mathbb{C}^4 \times \mathbb{P}^1$. Define the symplectic form $\omega_{\mathcal{X},\epsilon}$ on $\mathcal{X}$ by

$$\omega_{\mathcal{X},\epsilon} := (\omega_{\mathbb{C}^4} + \epsilon^2 \omega_{\mathbb{P}^1}) |_{\mathcal{X}}.$$
Let $B(\epsilon) = \{(y, z) \in \mathbb{C}^2 \mid |y|^2 + |z|^2 \leq \epsilon^2\} \subset \mathbb{C}^2$ be the ball of radius $\epsilon$. Consider the radial map $\rho_\epsilon : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus B(\epsilon)$,

$$\rho_\epsilon(y, z) = \frac{\sqrt{|y|^2 + |z|^2 + \epsilon^2}}{\sqrt{|y|^2 + |z|^2}}(y, z).$$

Let $\varrho_\epsilon = \text{id}_{\mathbb{C}^2} \times \rho_\epsilon : \mathbb{C}^2 \times (\mathbb{C}^2 \setminus \{0\}) \rightarrow \mathbb{C}^2 \times (\mathbb{C}^2 \setminus B(\epsilon))$. Then $\varrho_\epsilon$ preserves the conifold $\mathcal{Y}_0$ and it maps $\mathcal{Y}_0 \setminus \{0\}$ to $\mathcal{Y}_0(\epsilon) := \mathcal{Y}_0 \setminus (\mathcal{Y}_0 \cap (\mathbb{C}^2 \times B(\epsilon)))$.

McDuff-Salamon 98 $\implies$ the map

$$\psi_\epsilon := \varrho_\epsilon \mid_{\mathcal{Y}_0 \setminus \{0\}} \circ p \mid_{\mathcal{X} \setminus \mathbb{P}^1} : \mathcal{X} \setminus \mathbb{P}^1 \rightarrow \mathcal{Y}_0(\epsilon)$$

is a symplectomorphism.
Then we define the Lagrangian $L_{P,Q}$ in the resolved conifold $\mathcal{X}$ to be

$$L_{P,Q} := \psi_{e}^{-1}(\phi_{0}^{-1}(M_{K})).$$

Recall that $\phi_{0} : \mathcal{Y}_{0}\setminus\{0\} \to T^{*}S^{3}\setminus S^{3}$ is a symplectomorphism.

A nice property: consider the $S^{1}$-action on $\mathcal{X}$ defined as

$$u \cdot ((x, y, z, w), [s : t]) = ((u^{Q}x, u^{P}y, u^{-Q}z, u^{-P}w), [u^{-P-Q}s : t]).$$

Then $L_{P,Q}$ is preserved by the above action. We can use virtual localization techniques to study the open Gromov-Witten theory of $(\mathcal{X}, L_{P,Q})$. 

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Torus knots

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\[ L_{p,q} \cong S^1 \times \mathbb{R}^2 \]
A-model

\( \mathcal{X} \) The resolved conifold.

- \( L_{P,Q} \subset \mathcal{X} \) the Lagrangian sub-manifold constructed from the torus knot.
- Consider the open Gromov-Witten potential \( F_{g,n}^{(\mathcal{X},L_{P,Q})} \) of \( (\mathcal{X},L_{P,Q}) \).

Open Gromov-Witten invariants: Count maps \( f : C \to \mathcal{X} \), where \( C \) is a genus \( g \) bordered Riemann surface with \( n \) boundary circles such that \( f(\text{boundary circles}) \subset L_{P,Q} \).
A-model

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Q: Higher genus B-model?

- Chekhov-Eynard-Orantin 06 07: Topological recursion on spectral curves $\omega_{g,n}$ symmetric n-form.
- Brini-Eynard-Mariño 11 and Diaconescu-Shende-Vafa 11: conjecture that if we apply topological recursion to the mirror curve of $(\mathcal{X}, L_P, Q)$ $\rightarrow$ higher genus B-model.
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Brini-Eynard-Mariño 11 and Diaconescu-Shende-Vafa 11: conjecture that if we apply topological recursion to the mirror curve of $(X, L_P, Q)$ $\Rightarrow$ higher genus B-model.
Let $C \subset (\mathbb{C}^*)^2$ be a curve and let $X, Y$ be two meromorphic functions on $C$.

- Critical (ramification) points $P_\alpha$ of $X$: $dX = 0$.

- $X = e^{-x}$, $Y = e^{-y}$.

- Near each ramification point $P_\alpha$, use local coordinates:

\[
x = x_0 + \zeta_\alpha^2, \quad y = y_0 + \sum_{i=1}^{\infty} h_i^\alpha \zeta_i.
\]

- Near each ramification point, denote $\bar{p}$:

\[
\zeta_\alpha(\bar{p}) = -\zeta_\alpha(p).
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B-model

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Let \( C \subset (\mathbb{C}^*)^2 \) be a curve and let \( X, Y \) be two meromorphic functions on \( C \).

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B-model

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The initial data of the topological recursion is given by \( \omega_{0,1}, \omega_{0,2} \).

- \( \omega_{0,1} = ydx \).

Let the compactified mirror curve \( \bar{C} \) to be of genus \( g \). \( A_i, B_i \) are basis of \( H_1(\bar{C}; \mathbb{C}) \):

- \( A_i \cap B_j = \delta_{ij}, A_i \cap A_j = 0, B_i \cap B_j = 0. \)

Fundamental differential of the second kind (a.k.a. Bergmann kernel) \( \omega_{0,2}(p_1, p_2) \): symmetric 2-form on \( \bar{C} \times \bar{C} \). It is uniquely characterized by

\[
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Chekhov-Eynard-Orantin’s topological recursion

Chekhov-Eynard-Orantin construct symmetric forms $\omega_{g,n}$ on $C^n$:

- Initial data $\omega_{0,1} = ydx$, $\omega_{0,2}$ as above;
- The recursive algorithm is:

$$\omega_{g,n+1}(p_0, \ldots, p_n) = \sum_{P_\alpha} \text{Res}_{p \to P_\alpha} \frac{\int_P \omega_{0,2}(p_0, \cdot)}{2(y(p) - y(\bar{p}))dx(p)}$$

$$\cdot \left( \omega_{g-1,n+2}(p, \bar{p}, p_1, \ldots, p_n) \right)$$

$$+ \sum_{h=0}^{g} \sum_{A \cup B = \{1, \ldots, n\}, (h, |A|), (g-h, |B|) \neq (0,0)} \omega_{h,|A|+1}(p, \bar{p}_A) \omega_{g-h,|B|+1}(\bar{p}, \bar{p}_B).$$

$\omega_{g,n}$ is a symmetric $n$-form on $C^n$. 
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The mirror curve $C_q \subset (\mathbb{C}^*)^2$ of $\mathcal{X}$ is defined by the following equation

$$1 + U + V + qUV = 0$$

in $(\mathbb{C}^*)^2$. Here $q$ is a parameter on B-model. This curve allows a compactification into a genus 0 projective curve $\overline{C}_q$ in $\mathbb{P}^1 \times \mathbb{P}^1$, where $(1 : U)$ and $(1 : V)$ are homogeneous coordinates for each $\mathbb{P}^1$. 

$C_q \cong \mathbb{P}^1 \setminus \{4 \text{ pts}\}$
Mirror curve

Since \( \gcd(P, Q) = 1 \), we choose \( \gamma, \delta \in \mathbb{Z} \) (not uniquely) chosen such that

\[
\begin{pmatrix} Q & P \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z}).
\]

Consider the following change of variables

\[
X = U^Q V^P, \quad Y = U^\gamma V^\delta.
\]

We define the spectral curve as the quadruple

\[
(C_q \subset (\mathbb{C}^*)^2, \overline{C}_q \subset \mathbb{P}^1 \times \mathbb{P}^1, X, Y).
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The variables \( X, Y \) are holomorphic functions on \( C_q \) and meromorphic on \( \overline{C}_q \).
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\[ \omega_{0,2} = \frac{dU_1dU_2}{(U_1 - U_2)^2}. \]

The choice of \( \omega_{0,2} \) involves a symplectic basis on \( H_1(\overline{C}_q; \mathbb{C}) \) – since the genus of our \( \overline{C}_q \) is 0, this extra piece of datum is not needed.

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The mirror curve equation can be rewritten into

$$X = -V^P \left( \frac{V + 1}{1 + qV} \right)^Q.$$  

Let $\eta = X^{\frac{1}{Q}}$. Then $\eta$ is a local coordinate for the mirror curve $\overline{C}_q$ around $s_0 = (X, V) = (0, -1)$.

There exists $\delta > 0$ and $\epsilon > 0$ such that for $|q| < \epsilon$, the function $\eta$ is well-defined and restricts to an isomorphism

$$\eta : D_q \to D_\delta = \{ \eta \in \mathbb{C} : |\eta| < \delta \},$$

where $D_q \subset \overline{C}_q$ is an open neighborhood of $s_0$. Denote the inverse map of $\eta$ by $\rho_q$ and

$$\rho_q \times^n = \rho_q \times \cdots \times \rho_q : (D_\delta)^n \to (D_q)^n \subset (\overline{C}_q)^n.$$
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Mirror curve

We define

\[ W_{g,n}(\eta_1, \ldots, \eta_n, q) = \int_0^{\eta_1} \cdots \int_0^{\eta_n} (\rho_q^n)^* \omega_{g,n}, \quad 2g - 2 + n > 0. \]

Let \( f \in \mathbb{C}[[\eta_1, \ldots, \eta_n]] \). For a fixed integer number \( Q > 0 \), we denote

\[ h \cdot f(\eta_1, \ldots, \eta_n) = \sum_{k_1, \ldots, k_n=0}^{Q-1} \frac{f(a^{k_1}\eta_1, \ldots, a^{k_n}\eta_n)}{Q^n}, \]

where \( a \) is a primitive \( Q \)-th root of unity. This operation “throws away” all terms with degree not divisible by \( Q \).
Mirror symmetry

Theorem

Under the mirror map

$$F_{g,n}^{\chi,L_P,Q} = (-1)^{g-1+n} Q^n (\eta \cdot W_{g,n}) (\eta_1, \ldots, \eta_n, q).$$

In other words, $F_{g,n}^{\chi,L_P,Q}$ is equal to the part in the power series expansion of $(-1)^{g-1+n} Q^n W_{g,n}(q, \eta_1, \ldots, \eta_n)$ whose degrees of each $\eta_k$ are divisible by $Q$. 
Some remarks

**Remark**

The above theorem is an all genus open-closed mirror symmetry between the Gromov-Witten theory of $(X, L_K)$ and the topological recursion of the mirror curve.

On the other hand, Borot-Eynard-Orantin 13 $\Longrightarrow$ Topological recursion is equivalent to the colored HOMFLY polynomial of the knot $K$.

Therefore, the following three objects are equivalent:

1. the open-closed Gromov-Witten invariants of $(X, L_K)$;
2. the Eynard-Orantin invariants of the mirror curve;
3. the colored HOMFLY polynomial of the knot $K$.

The equivalence of colored HOMFLY and GW is called large $N$ duality.
Some remarks

When $P = Q = 1$, the Lagrangian $L_{1,1}$ is called an **Aganagic-Vafa brane**. The large $N$ duality for $(\mathcal{X}, L_{1,1})$ is also called the **Mariño-Vafa formula**. An Aganagic-Vafa brane $\mathcal{L}$ can be defined for any toric Calabi-Yau 3-folds/3-orbifolds $\widetilde{\mathcal{X}}$.

Bouchard-Klemm-Marino-Pasquetti 07, 08: Introduce the Remodeling Conjecture

**Theorem (Remodeling Conjecture)**

*If we expand $\omega_{g,n}$ under suitable local coordinate on the mirror curve $C$ of $\widetilde{\mathcal{X}}$, we obtain the open Gromov-Witten potential $F_{g,n}(\widetilde{\mathcal{X}}, \mathcal{L})$ under the mirror map.*
Remodeling Conjecture

- When \( \tilde{\mathcal{X}} = \mathbb{C}^3 \), open part: L. Chen, J. Zhou; closed part: Bouchard-Catuneanu-Marchal-Sulkowski, S. Zhu.
- When \( \tilde{\mathcal{X}} \) is smooth: Eynard-Orantin.
- General semi-projective toric CY 3-orbifolds: Fang-Liu-Z.

If we start from \((\mathcal{X}, L_{1,1})\), then the Remodeling conjecture and the topic today can be viewed as generalizations along two different directions:

- Remodeling conjecture: generalizes the ambient space: resolved conifold \( \leadsto \) toric CY 3-orbifolds.
- Today’s topic: generalizes the Lagrangian sub-manifold: \( L_{1,1} \leadsto L_{P,Q} \) (trivial knot \( \leadsto \) torus knots).
Thank you!