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Some analogues of function theoretic objects and stochastic processes on infinite graphs

Atsushi Atsuji, Keio Univ.

based on a joint work with Hiroshi Kaneko (Tokyo Univ. of Science)

Outline :

In this talk we discuss some discrete analogues of classical function theory in the context of tropical mathematics. As the objects in classical function theory, we mainly consider Nevanlinna theory for the value distribution of meromorphic functions.

§1. Classical Nevanlinna theory

§2. One dimensional tropical Nevalinna theory

§2-1. Original setting

§2-2. A probabilistic interpretation and extension

§3. Diffusion process on infinite graphs

§4. Nevalinna theory on graphs

§5. Remarks and future works

1 Classical Nevanlinna theory

Nevanlinna theory studies the value distribution of meromorphic functions quantitatively.

Classical Qualitative results:

Liouville theorem : No non-constant bounded holomorphic functions on $\mathbb{C}$.

Picard (little) theorem : Any non-constant meromorphic function on $\mathbb{C}$ can omit at most two values in $\mathbb{C} \cup \{\infty\}$.

Picard (big) theorem : Any meromorphic function on a neighborhood of its essential singularity takes all values of $\mathbb{C} \cup \{\infty\}$ infinitely often except for two values.

Nevanlinna theory is an quantitative extension of Picard's theorem. (R. Nevalinna, 1926)

Setting by R. Nevanlinna (1926).

cf. Hayman, W. K. : Meromorphic functions (1964)

f: non-constant meromorphic function on $\mathbb{C}$.

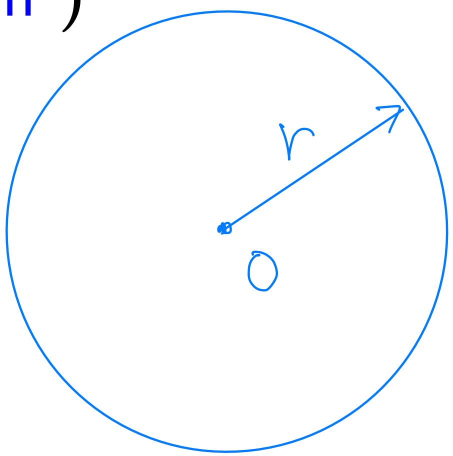
a_μ : zeros of f, b_ν : poles of f. $a \in \mathbb{C} \cup \{\infty\}$.

[Nevanlinna functions]

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

$$m(r, a) := m(r, \frac{1}{f-a}),$$ (proximity function)

$$m(r, f) = m(r, \infty).$$



$$n(r,f) := \sum_{|b_\nu|<r} \text{(counting with multiplicity)},$$

$$N(r,f) := \int_0^r n(t,f)\frac{dt}{t} = \sum_{|b_\nu|<r} \log\frac{r}{|b_\nu|}$$

$$N(r,\frac{1}{f}) := \int_0^r n(t,\frac{1}{f})\frac{dt}{t} = \sum_{|a_\mu|<r} \log\frac{r}{|a_\mu|}$$

$$N(r,a) := N(r,\frac{1}{f-a}) \text{ (counting function).}$$

$$N(r,f) = N(r,\infty), \quad N(r,\frac{1}{f}) = N(r,0).$$

$$T(r,f) := m(r,f) + N(r,f) \ \text{(characteristic function of } f\text{)}$$

Nevanlinna theory consists of two "Main theorems".

Theorem 1 (First main theorem of Nevanlinna theory, FMT)

$$m(r,a) + N(r,a) = T(r,f) + O(1).$$

Theorem 2 (Second main theorem of Nevanlinna theory, SMT)

$a_1, \ldots, a_q \in \mathbb{C} \cup \{\infty\}$: distinct points.

$$\sum_{j=1}^{q} m(r,a_j) + N_1(r) \leq 2T(r,f) + O(\log T(r) + \log r),$$

holds for $r \notin E \subset [0,\infty)$ with $\int_E dr < \infty$, where

$$N_1(r) = N(r,1/f') + 2N(r,f) - N(r,f') \geq 0.$$

It is known that

- f : non-constant $\Rightarrow T(r.f) \to \infty$ $(r \to \infty)$.

Define the defect of f:

$$\delta_f(a) := \liminf_{r\to\infty} \frac{m(a,r)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(a,r)}{T(r,f)}$$

FMT implies $0 \le \delta_f(a) \le 1$.

- f : rational $\Leftrightarrow T(r,f) = O(\log r)$.

$\rightsquigarrow$ f : non-rational (transcendental) $\Rightarrow T(r,f)/\log r \to \infty$

$\Rightarrow O(\log T(r,f) + \log r) = o(T(r))$ in the RHS of SMT.

$\rightsquigarrow$

$$\sum_{k=1}^{q} \delta_f(a_k) \le 2 \quad \text{(defect relation).}$$

If f omits a, $\delta_f(a) = 1$. $\rightsquigarrow$ Picard's little theorem.

[Proof of SMT]

By direct calculation

Lemma 3

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) - 2T(r, f) \leq S(r),$$

$$S(r) = m(r, \frac{f'}{f}) + m(r, \sum_{a_j \neq \infty} \frac{f'}{f - a_j}) + O(1).$$

The remainder term $S(r)$ can be estimated by using the following estimate. This is the key point of the proof.

Theorem 4 (Lemma of logarithmic derivative (LLD))

$$m(r, \frac{f'}{f}) \leq O(\log T(f,r) + \log r)$$

holds for r except a set of finite Lebesgue measure.

We will consider a tropical version of this theorem. $\leftarrow$ OUR AIM.

Rem. There is some relationship between the original Nevanlinna theory and Probability theory. Interpretation in the language of complex Brownian motion and stochastic calculus. (Carne '86, A.'95).

2 One dimensional tropical Nevanlinna theory

2.1 Original version

We first look at the setting as the following prior works:

- Laine, I. and Tohge, K. : Tropical Nevanlinna theory and second main theorem. Proc. London Math. Soc, 2009.
- Halburd, R.G. and Southall, N. : Tropical Nevanlinna theory and ultra-discrete equations, Int. Math. Res. Notices.2009(2009)

[Tropical arithmetic operations]

$$x \oplus y := \max\{x, y\}, \quad x \otimes y := x + y, \quad x \oslash y := x - y, \quad x^{\otimes \alpha} := \alpha x$$

unit element

$$0_o := -\infty, \quad 1_o := 0$$

Rem. These operations follow from usual operations after ultra discrete limit procedure.

[Tropical polynomials]

$$f(x) = \bigoplus_{j=0}^{p} a_j \otimes x^{\otimes s_j} = a_0 \otimes x^{\otimes s_0} \oplus a_1 \otimes x^{\otimes s_1} \oplus \cdots \oplus a_p \otimes x^{\otimes s_p}$$

$(s_0 < s_1 < \cdots < s_p,\ a_j \in \mathbb{R}\ (j = 1, \ldots, p))$.

$\Leftrightarrow$

$$f(x) = \max\{a_0 + s_0 x,\ a_1 + s_1 x, \ldots, a_p + s_p x\}.$$

$\rightsquigarrow$ f : a piecewise linear, convex function.

[Tropical rational functions]

$$P(x) \oslash Q(x) = P(x) - Q(x)\ (P, Q : \text{ tropical poly. }).$$

[zeros, poles]

Let f: piecewise linear, $f'_+(x)$: right derivative, $f'_-(x)$: left derivative.

$$w_f(x) := f'_+(x) - f'_-(x).$$

$w_f(x) > 0 \Rightarrow x$: zero, $w_f(x) < 0 \Rightarrow x$: pole. $|w_f(x)|$: its multiplicity.

[Tropical meromorphic function]

They define

a tropical meromorphic function on $\mathbb{R}$

$:=$ a piecewise linear function on $\mathbb{R}$

i.e. a tropical meromorphic function is locally a tropical rational function.

We will extend this class of tropical meromorphic functions in our setting.

[Basic analogy between classical and tropical function theory]

In this talk, we say a point x is a convex (resp. concave) point of f if f is strictly convex (resp. concave) on a neighborhood of x.

classical	$\leftrightarrow$	tropical
complex plane $\mathbb{C}$	$\leftrightarrow$	$\mathbb{R}$ line
a disc $\{\lvert z\rvert < r\}$	$\leftrightarrow$	an interval $(-r, r)$
meromorphic function f	$\leftrightarrow$	piecewise linear function f ($\to$ δ-convex)
zero	$\leftrightarrow$	convex point $\vee$,
pole	$\leftrightarrow$	concave point $\wedge$
divisor	$\leftrightarrow$	points with (integer) coefficients
$\log \lvert f\rvert$	$\leftrightarrow$	f

One may be convinced of the last four correspondences by the following formulas:

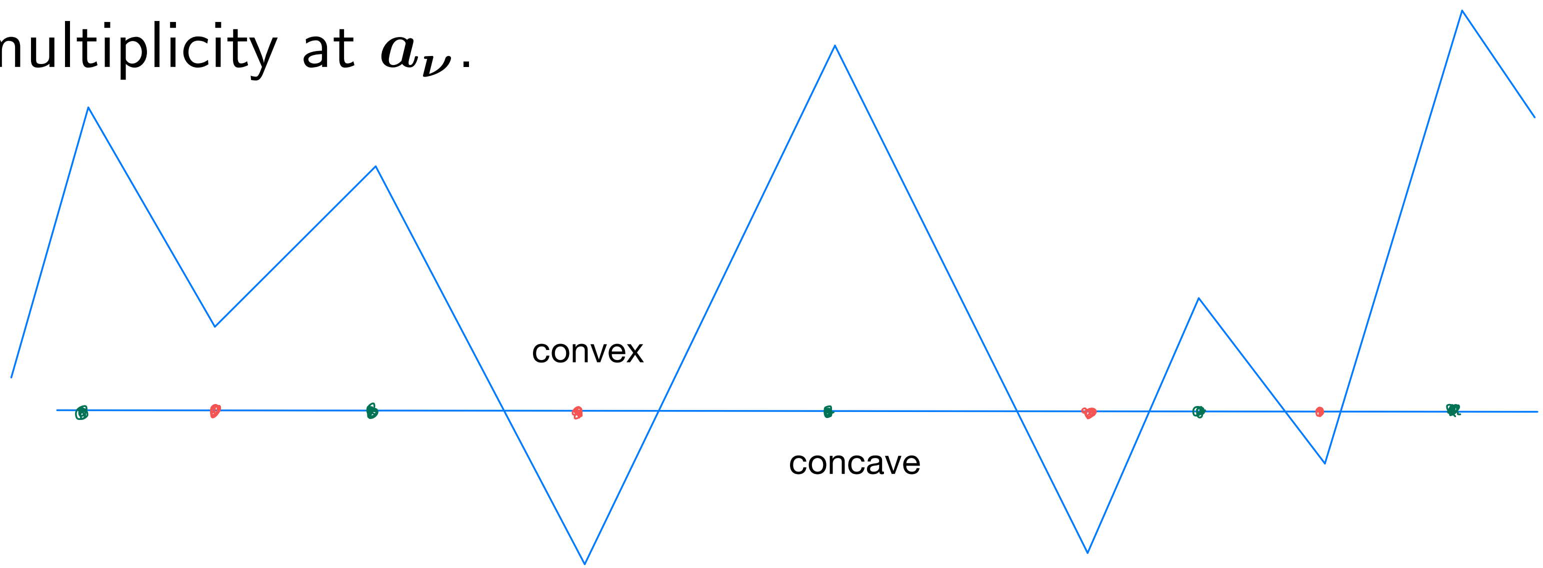
$$\text{classical } \Delta \log |f| = \sum_{\nu} \tau_{\nu}(a_{\nu})\delta_{a_{\nu}} - \tau_{\nu}(b_{\nu})\delta_{b_{\nu}}$$

$$: \ (a_{\nu} : \text{ zero }, b_{\nu} : \text{ pole })$$

$$\text{tropical } f'' = \sum_{\nu} \tau_{\nu}(a_{\nu})\delta_{a_{\nu}} - \tau_{\nu}(b_{\nu})\delta_{b_{\nu}}$$

$$: \ (a_{\nu} : \text{ convex }, b_{\nu} : \text{ concave}).$$

$\tau_{\nu}(a_{\nu})$: multiplicity at a_{ν}.



[Tropical Nevanlinna functions]

$r > 0$. $\{b_\nu\}$: poles of f, $\tau_f(b_\nu)$: multiplication.

$$m(r, f) := \frac{1}{2}\{f^+(r) + f^+(-r)\}$$

$$N(r, f) := \frac{1}{2}\sum_{|b_\nu|<r} \tau_f(b_\nu)(r - |b_\nu|)$$

$$T(r, f) := m(r, f) + N(r, f).$$

- f : tropical rational function

$\Leftrightarrow T(r, f) = O(r)$ and

$\inf\{\tau_f(b_\nu), \tau_f(a_\nu) \mid a_\nu :$ zero, $b_\nu :$ pole$\} > 0$.

2.2 probabilistic interpretation and extension

[δ- convex function]

Tropical polynomial ;

$$f(x) = \max\{a_0 + s_0 x,\ a_1 + s_1 x, \ldots, a_p + s_p x\}.$$

$\{a_k, s_k\}$: countably infinite $\Rightarrow f$: convex.

tropical rational function $=$ a difference of tropical polynomials.

Definition 5 A continuous function f on $\mathbb{R}$ is a δ- convex function on $\mathbb{R}$ if f can be expressed locally by a difference of two convex functions. All the δ- convex functions on $\mathbb{R}$ is denoted by $\mathcal{A}(\mathbb{R})$.

$f \in \mathcal{A}(\mathbb{R})$

$\Rightarrow \exists f''(da)$: signed measure on any compact set,

$$f''(da) = (f'')^+(da) - (f'')^-(da).$$

(Hahn-Jordan decomp.)

f : tropical meromorphic

$\Rightarrow$

$$(f'')^+(da) = \sum_j \tau_f(a_j)\delta_{a_j}, \ (f'')^-(da) = \sum_j \tau_f(b_j)\delta_{b_j}.$$

The prior works mentioned above used the following "Poisson-Jensen formula".

[one-dimensional Poisson-Jensen formula]

f : tropical meromorphic function on $(-r, r)$, $\{a_j\}$: zero, $\{b_j\}$: pole.

$$f(x) = \frac{x+r}{2r} f(r) + \frac{r-x}{2r} f(-r) - \frac{1}{2} \sum_{|a_j|<r} \tau_f(a_j) \{ \frac{1}{r}(r^2 - a_j x) - |x - a_j| \} + \frac{1}{2} \sum_{|b_j|<r} \tau_f(b_j) \{ \frac{1}{r}(r^2 - b_j x) - |x - b_j| \}.$$

This formula can be generalized by Itô-Tanaka formula as follows.

[Itô-Tanaka formula] Let $f \in \mathcal{A}(\mathbb{R})$ and X Brownian motion on $\mathbb{R}$.

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da),$$

where L_t^a is the local time of X at a.

Set $\tau_r := \inf\{t > 0 : |X_t| \geq r\}$ for $r > 0$. For $x \in (-r, r)$

$$E_x[f(X_{\tau_r})] = f(x) + \frac{1}{2}\int_{-r}^{r} E_x[L_{\tau_r}^a]f''(da).$$

Apply Itô-Tanaka to $f(x) = |x|$.

$$|X_t - a| = |X_0 - a| + \int_0^t \mathrm{sgn}(X_s - a)dX_s + L_t^a \text{ (Tanaka formula).}$$

$$E_x[L^a_{\tau_r}] = E_x[|X_{\tau_r} - a|] - |x - a|.$$

$$P_x(X_{\tau_r} = r) = \frac{x + r}{2r}, \; P_x(X_{\tau_r} = -r) = \frac{r - x}{2r}.$$

For $a \in (-r, r)$

$$E_x[L^a_{\tau_r}] = (r-a)\frac{x + r}{2r} + (r+a)\frac{r - x}{2r} - |x-a| = \frac{1}{r}(r^2 - ax) - |x-a|.$$

Combining these formulas, we obtain 1-dim P-J formula.

Define Nevanlinna functions for δ-convex fuctions.

$$
\begin{aligned}
m_x(r,f) &:= E_x[f(X_{\tau_r})] = \frac{x+r}{2r} f(r) + \frac{r-x}{2r} f(-r), \\
N_x^Z(r,f) &:= \frac{1}{2}\int_{-r}^{r} E_x[L^a_{\tau_r}](f'')^+(da) \\
&= \frac{1}{2}\int_{-r}^{r} \left(\frac{1}{r}(r^2 - ax) - |x-a|\right)(f'')^+(da) \\
N_x^P(r,f) &:= \frac{1}{2}\int_{-r}^{r} E_x[L^a_{\tau_r}](f'')^-(da) \\
&= \frac{1}{2}\int_{-r}^{r} \left(\frac{1}{r}(r^2 - ax) - |x-a|\right)(f'')^-(da).
\end{aligned}
$$

$$
m(r,f) := m_0(r, f \vee 0),\ N(r,f) := N_0^P(r,f),
$$

$$
T(r,f) := m(r,f) + N(r,f).
$$

[Tropical LLD]

Shifted meromorphic function f_c defined by $f_c(x) := f(x+c)$ for $c \in \mathbb{R}$.

R. G. Halburd and N. J. Southall gave a tropical analogue of the lemma on logarithmic derivative in classical Nevanlinna theory.

For $\forall \epsilon > 0$ $\exists E_\epsilon$ s.t.

$$m(r, f_c - f) \leq \frac{14 \cdot 2^{1+\epsilon}|c|}{r}\{T(r+|c|, f)^{1+\epsilon} + o(T(r+|c|, f))\}$$

holds for $r \notin E_\epsilon$ with $|E_\epsilon| < \infty$.

Moreover, if $T(r)$ has polynomial growth, we have

$$m(r, f_c - f) \leq o(T(r)).$$

Rem. We can show this LLD under the above probabilistic setting for δ-convex functions.

Based on these basic observations, we wish to give an extension to the case of infinite graph case. One may say that it is a higher dimensional extension of one-dimensional tropical Nevanlinna theory.

Extension: $\mathbb{R}$ can be regarded a line graph.

$$\mathbb{R} \ \leftrightarrow \ \boldsymbol{G = (V, E)}$$

$$\text{Brownian motion} \quad \leftrightarrow \quad \text{a diffusion process on } \boldsymbol{G}.$$

3 Diffusion process on graphs

[Setting on graphs]

$G = (V, E)$: connected, locally finite, i.e., $\#N(x) < \infty$ ($\forall x \in V$),
($N(x) := \{y \in V \mid \{x, y\} \in E\}$).

Conductance $C_{x,y}$ ($x, y \in V$) :

$$\begin{cases} C_{x,y} > 0 & \text{if } \{x, y\} \in E, \\ C_{x,y} = 0 & \text{if } \{x, y\} \notin E \end{cases}$$

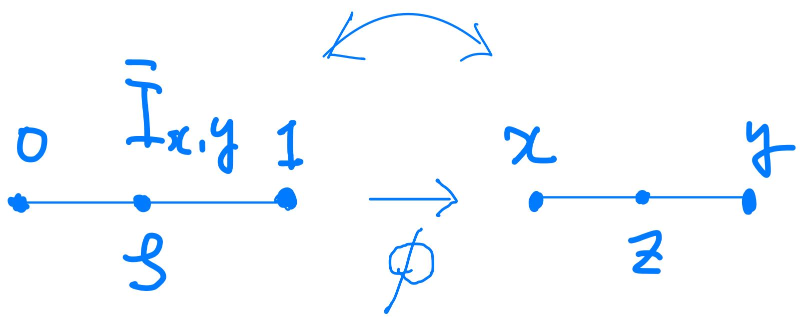
and $C_{x,y} = C_{y,x}$.

[Canonical distance on graphs]

We take a natural distance on G to make G a continuum which is a metric space.

Edges to be continuum:

$x \leftrightarrow 0 \in \overline{I}_{x,y}$, $y \leftrightarrow 1 \in \overline{I}_{x,y}$.



$\zeta \in \overline{I}_{x,y} (= [0,1]) \ \leftrightarrow z \in \{x, y\}$. $\Rightarrow \ d(z, x) = \zeta$ ($\Leftrightarrow$ $d(z, y) = 1 - \zeta$).

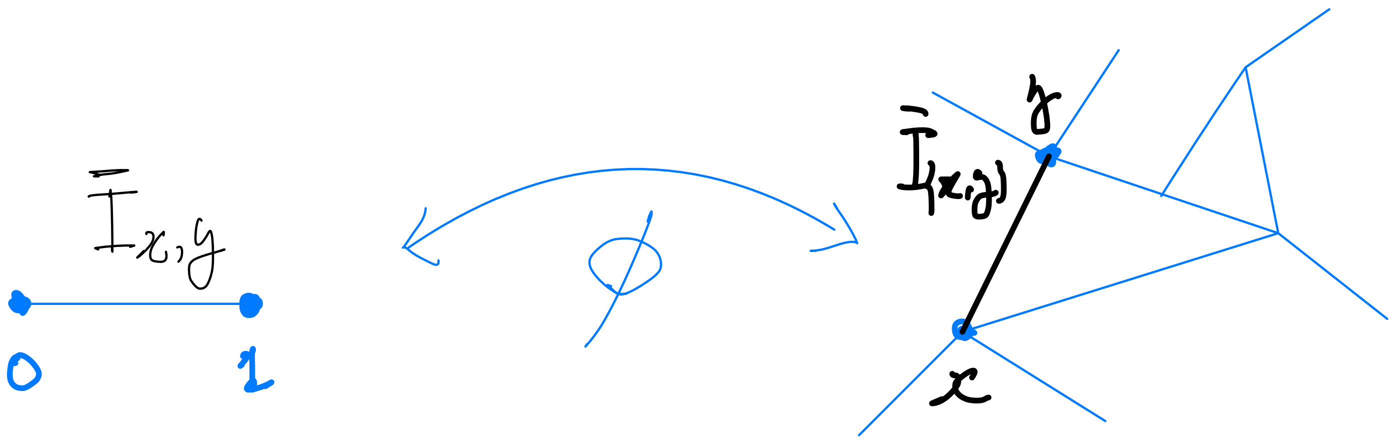
The set of the points between x, y is denoted by $\overline{I}_{\{x,y\}}$ and the map $\phi : \overline{I}_{x,y} \to \overline{I}_{\{x,y\}}$ is given by $\phi(z) = \zeta$.

$I_{\{x,y\}} := \overline{I}_{\{x,y\}} \setminus \{x, y\}$.

$\overline{I} = \cup_{\{x,y\} \in E} \overline{I}_{\{x,y\}}$, distance $d(x, y)$ with $x, y \in \overline{I}$ by

$d(x, y) = \min\{d(x, x_0) + \sum_{i=0}^{n-1} d(x_i, x_{i+1}) + d(x_n, y) \mid x_0, x_1, \ldots, x_n \in V$ with $\{x_i, x_{i+1}\} \in E \quad (i = 0, 1, \ldots, n-1)\}$.

Then $\overline{I}$ is viewed as a topological space which admits the topology induce by the metric d and the family of continuous function on $\overline{I}$ is denoted by $\mathcal{C}(\overline{I})$.



[Dirichlet form]

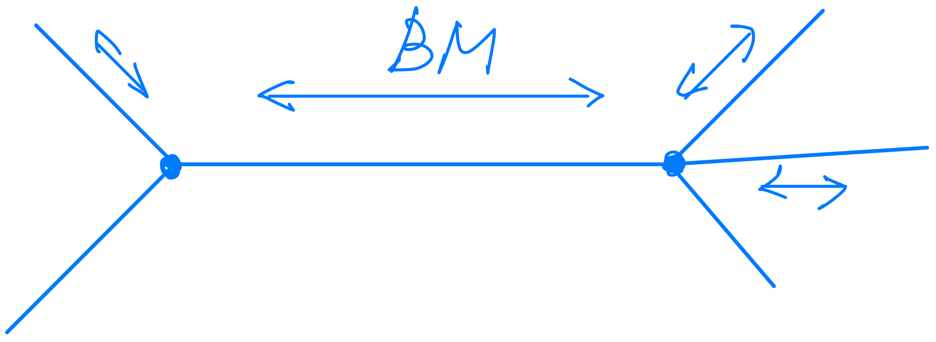
Intuitively, we construct a diffusion process $\boldsymbol{X}$ on $\overline{\boldsymbol{I}}$ such that

- $\boldsymbol{X}$ behaves as a Brownian motion with time-change defined by the conductance on each open edge.
- When $\boldsymbol{X}$ arrives at a vertex, simultaneously $\boldsymbol{X}$ chooses an edge out of the vertex and goes on the edge like a reflecting Brownian motion.

It looks like a Walsh Brownian motion around vertices.

(Barlow-Pittman-Yor (1989), Freidlin-Sheu (2000) : SDE methods for diffusion processes on graphs)

We construct such a diffusion process on $\overline{\boldsymbol{I}}$ by the Dirichlet form method.



u : real-valued function on $I_{\{x,y\}}$.

$u \in H^1(I_{\{x,y\}}) \ \Leftrightarrow \ u \circ \phi \in H^1(I_{x,y})$,

i.e., $u \circ \phi \in L^2(I_{x,y})$ and $(u \circ \phi)' \in L^2(I_{x,y})$, where

$I_{x,y} = \overline{I}_{x,y} \setminus \{0, 1\}$.

$$\|u\|_{H^1} = \Big(\sum_{\{x,y\} \in E} (\|u \circ \phi\|^2_{L^2(I_{x,y})} + C_{x,y} \|(u \circ \phi)'\|^2_{L^2(I_{x,y})}) \Big)^{1/2}$$

$\mathcal{F} = \{u \in \mathcal{C}(\overline{I}) \mid \ u|_{I_{x,y}} \in H^1(I_{\{x,y\}}) \ \forall \{x, y\} \in E$ and $\|u\|_{H^1} < \infty\}$

Dirichlet form $\mathcal{E}$:

$$\mathcal{E}_{x,y}(u, v) = C_{x,y}((u \circ \phi)', (v \circ \phi)')_{L^2(I_{x,y})} \text{ for } \{x, y\} \in E,$$

$$\mathcal{E}(u, v) = \sum_{\{x,y\} \in E} \mathcal{E}_{x,y}(u, v)$$

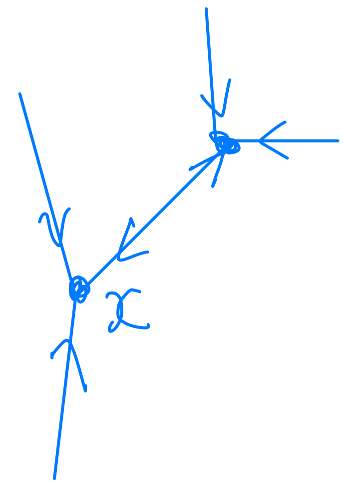
for any $u, v \in \mathcal{F}$.

Green's formula :

$$\mathcal{E}_{x,y}(u,v)$$
$$= -C_{x,y}\frac{\partial v}{\partial r_{x,y}}u(x) - C_{x,y}\frac{\partial v}{\partial r_{y,x}}u(y) - C_{x,y}(v'', u)_{L^2(I_{x,y})},$$

where

$$\frac{\partial v}{\partial r_{x,y}} := \lim_{\zeta\downarrow 0, \zeta\in I_{x,y}} \frac{v(\phi(\zeta)) - v(\phi(0))}{\zeta},$$
$$\frac{\partial v}{\partial r_{y,x}} := \lim_{\zeta\uparrow 1, \zeta\in I_{x,y}} \frac{v(\phi(\zeta)) - v(\phi(1))}{\zeta - 1}.$$



There exist signed measures $\nu_1^{[v]}$ on $I_{\{x,y\}}$ and $\nu_2^{[v]}$ on $\overline{I}_{\{x,y\}} \setminus I_{\{x,y\}}$ such that $u \in \mathcal{F} \cap \mathcal{C}(\overline{I}_{\{x,y\}})$ and v is "good" implies

$$\mathcal{E}(u,v) = -C_{x,y}\int_{I_{\{x,y\}}} u d\nu_1^{[v]} - C_{x,y}\int_{\overline{I}_{\{x,y\}}\setminus I_{\{x,y\}}} u d\nu_2^{[v]}.$$

$$\mathcal{C}_0^1 = \{f \in \mathcal{C}(\overline{I}) \mid \mathbf{supp}[f] : \text{ bounded w.r.t } d$$

$$\text{and } f \circ \phi|_{I_{x,y}} \text{ admits the first order derivative on each} I_{x,y}$$

$$\text{with } \lim_{\zeta \downarrow 0} (f \circ \phi|_{I_{x,y}})'(\zeta) = \frac{\partial v}{\partial r_{x,y}}$$

$$\text{and } -\lim_{\zeta \uparrow 1} (f \circ \phi|_{I_{x,y}})'(\zeta) = \frac{\partial v}{\partial r_{y,x}}\}.$$

Proposition 6 $\mathcal{C}_0^1$ is dense in $\mathcal{C}(\overline{I})$ with respect to the uniform norm and so is in $\mathcal{F}$ with respect to the H^1-norm as well.

$\rightsquigarrow$ $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form.

Obviously $(\mathcal{E}, \mathcal{F})$ is local.

Then $(\mathcal{E}, \mathcal{F})$ defines a diffusion process (X_t, P_x) on $\bar{I}$.

4 Nevanlinna theory on graphs

[Assumption on graphs : Tree structure and geodesics]

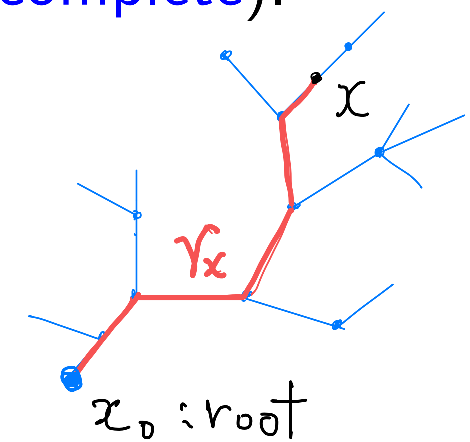
Take a reference point $x_0 \in V$ (root).

Assume a tree structure on G as follows.

We assume that for any $x \in V$ there exists a unique continuous path γ_x from $[0, d(x_0, x)]$ to $\overline{I}$ and γ_x satisfies that $\gamma_x(0) = x_0$ and $\gamma_x(d(x_0, x)) = x$. We call γ_x a geodesic joining x and x_0.

We assume that $\gamma_x(t)$ $(t \in [0, d(x_0, x)])$ can be extended to $\gamma(t)$ $(t \in [0, \infty))$ such that $d(x_0, \gamma(t)) \to \infty$ $(t \to \infty)$ and γ is the geodesic joining $\gamma(t)$ and x_0 for $t \in (0, \infty)$ (geodesically complete).

We also call such γ a geodesic on G.



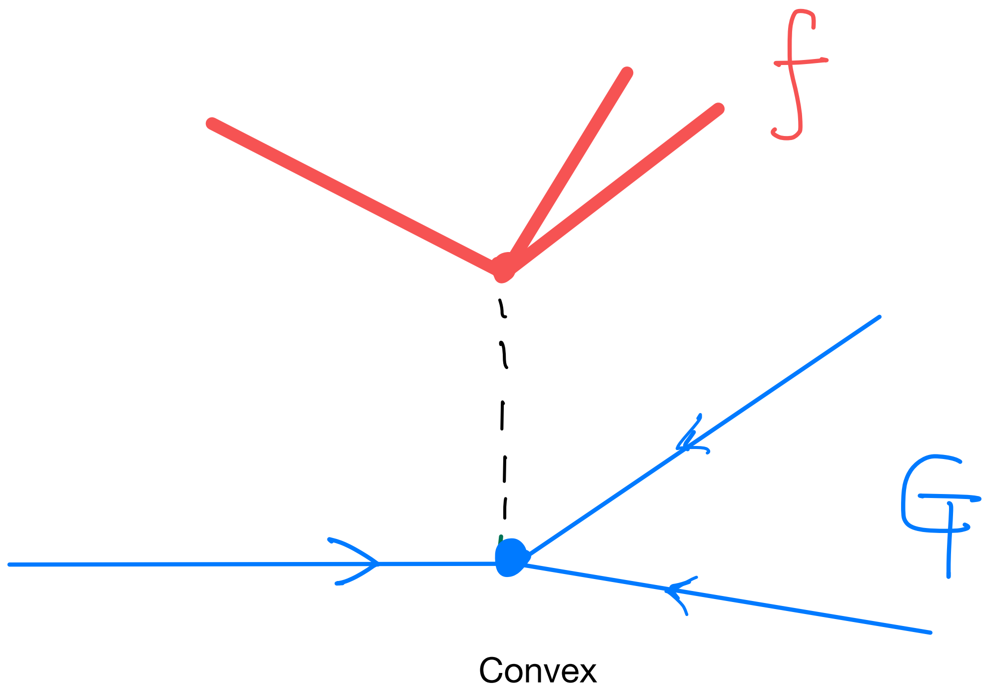
[δ-convex functions on $\overline{I}$]

Definition 7 i) We say a continuous function f on $\overline{I}$ is a convex function on an open set $U \subset \overline{I}$ if $f \circ \gamma$ is a convex function on J for any geodesic γ and any open subinterval J in the set $\{t \in [0, \infty) : \gamma(t) \in U\}$.

ii) We say a continuous function f on $\overline{I}$ is a δ-convex function on $\overline{I}$ if there exist convex functions $f_{1,R}$ and $f_{2,R}$ such that $f = f_{1,R} - f_{2,R}$ on $B(R)$ for each $R > 0$ where $B(R) := \{x \in \overline{I} | d(x_0, x) < R\}$. We denote the set of all δ-convex functions on $\overline{I}$ by $\mathcal{A}(\overline{I})$.

iii) We call $f \in \mathcal{A}(\overline{I})$ a tropical meromorphic function on $\overline{I}$ if $f \circ \gamma \in \mathcal{A}_1([0, \infty))$ for any geodesic γ. We denote the set of all tropical meromorphic functions on $\overline{I}$ by $\mathcal{A}_1(\overline{I})$.

- If $v \in V$ is a convex point of f, it means that f is convex at v in any direction.
- If f is δ-convex, ii) implies that any $v \in V$ is either convex point or concave one for f.
- $f \circ \gamma$ is convex $\Rightarrow (f \circ \gamma)''$ is a measure.



We can see that $u \in \mathcal{F}_{\mathrm{loc}}$ if $u \in \mathcal{A}(\bar{I})$. Then for $u \in \mathcal{A}(\bar{I})$, we have a Fukushima decomposition:

$$u(X_t) - u(X_0) = M_t^{[u]} + A_t^{[u]} + l_t^{[u]},$$

where $M_t^{[u]}$ is a local martingale additive functional, and $A_t^{[u]}$ and $l_t^{[u]}$ are additive functionals of zero energy.

$A_t^{[u]}$ corresponds to a signed smooth measure $\nu_1^{[u]}$ supported on edges, $l_t^{[u]}$ to $\nu_2^{[u]}$ supported on vertices.

$\nu_2^{[u]} = (\sum_{y \in N(x)} C_{x,y} \frac{\partial u}{\partial r_{x,y}})\delta_x$ on a small neighborhood of $x \in V$.

Hahn-Jordan decomposition;

$$\nu_1^{[u]} = \nu_1^{[u],+} - \nu_1^{[u],-},\ \nu_1^{[u]} = \nu_2^{[u],+} - \nu_2^{[u],-}.$$

$$A_t^{[f],+} \ \leftrightarrow \ \nu_1^{[u],+}, \quad A_t^{[f],-} \ \leftrightarrow \ \nu_1^{[u],-}$$
$$(l_t^{[f]})^+ \ \leftrightarrow \ \nu_2^{[u],+}, \quad (l_t^{[f]})^- \ \leftrightarrow \ \nu_2^{[u],-}.$$

[Nevanlinna functions]

Define $T_r := \inf\{t > 0 \ : \ X_t \notin B(r)\}$,

$$m_x(r, f) := E_x[f(X_{T_r})],$$
$$N_x^Z(r, f) := E_x[A_{T_r}^{[f],+}] + E_x[(l_{T_r}^{[f]})^+],$$
$$N_x^P(r, f) := E_x[A_{T_r}^{[f],-}] + E_x[(l_{T_r}^{[f]})^-].$$

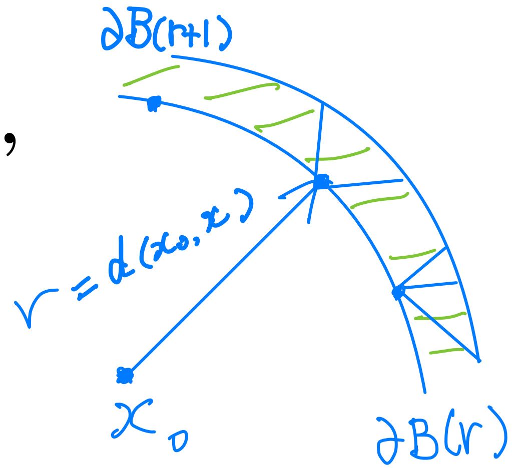
$$m(r, f) := m_{x_0}(r, f \vee 0), \ N(r, f) := N_{x_0}^P(r, f)$$
$$T(r, f) := m(r, f) + N(r, f).$$

$$\rightsquigarrow \quad T(r, f) = N_{x_0}^Z(r, f) + m_{x_0}(r, (-f) \vee 0) + f(x_0) \text{ (FMT)}.$$

To show LLD under this setting, we assume the following isotropic conditions of metrical property G;

(A1)

$C_{x,y}$ is depending only on $d(x_0, x)$ provided that $d(x_0, x) = d(x_0, y) - 1$



(A2) $\mathbf{deg}(x)$ is depending only on $d(x_0, x)$,

where $\mathbf{deg}(x)$ denotes the degree at $x \in V$.

$$\mathcal{A}_{\#}(\overline{I}) := \{u \in \mathcal{A}(\overline{I}) \ : \ \mathbf{Supp}\,[(u \circ \gamma)''_S] \text{ is discrete in } \overline{I} \text{ for any geodesic } \gamma\},$$

where $(u \circ \gamma)''_S$ denotes the singular part of $(u \circ \gamma)''$ with respect to Lebesgue measure. $\mathcal{A}_1(\overline{I}) \subset \mathcal{A}_{\#}(\overline{I})$.

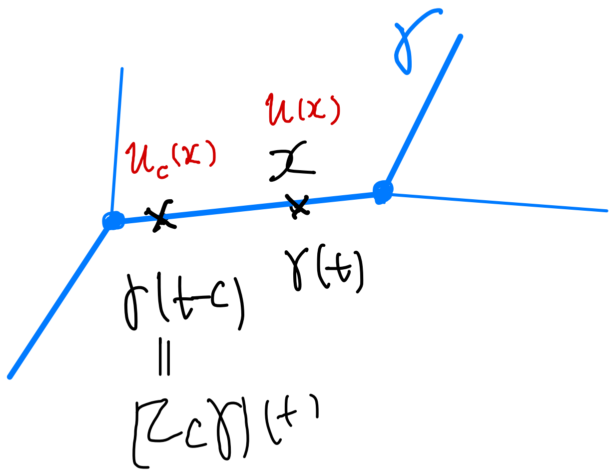
We now introduce a shift of functions on the graphs.

For any c with $0 < c \leq 1$, a translation $\tau_c : t \mapsto t - c$ on $\mathbb{R}$ naturally acts on geodesics as

$$(\tau_c \gamma)(t) := \gamma(t - c) \ (t \geq c),$$

and the shift action $u \mapsto u_c$ on $\mathcal{A}(\overline{I})$ is defined by

$$u_c(x) := \begin{cases} u((\tau_c \gamma_x)(d(x_0, x))) & (d(x_0, x) \geq c), \\ u(x_0) & (d(x_0, x) < c). \end{cases}$$



Theorem 8 Assume (A1) and (A2). Let $0 < c \leq 1$ and $f \in \mathcal{A}_{\#}(\overline{I})$. For any $\epsilon > 0$, there exists a set $E_\epsilon \subset [0, \infty)$ of finite logarithmic measure such that

$$m(r, f_c - f) \leq \frac{C(r)}{r}\{T(r+1, f)^{1+\epsilon} + O(1)\}$$

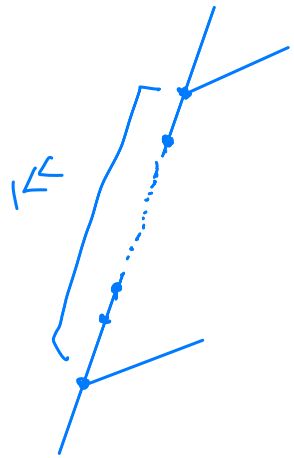
holds for $r \notin E_\epsilon$, where $C(r)$ is a positive function depending only on ϵ, c and G, independent of f.

We have to know the size of $C(r)$ from geometric conditions of G.

$k_n := (\deg(\gamma(n)) - 1)^{-1}$ (cf. (A2)).

Proposition 9 Assume the degrees of G on vertices satisfy

$$\alpha := \liminf_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \log k_j > -1.$$



If $T(r, f)$ has polynomial growth, then

$$m(r, f_c - f) = o(T(r)) \ (r \to \infty).$$

Rem. Under the above assumption, $C(r) \sim r^{-\alpha+\delta}$ with $-\alpha + \delta < 1$ for some $\delta > 0$.

If G has finite number of the ends and all ends are line graph, then $\alpha = 0$.

5 Remarks and future works

1) Although we do not have applications like Picard's theorem as in classical case, one-dimensional tropical Nevanlinna theory can be applied to study of ultra-discrete equations (See Halburd-Southall(2009)).

2) In this talk we treat Nevanlinna theory only for $\mathbb{R}$-valued functions. It is natural to ask how about the values takes place in higher dimensional targets. There are some works on "tropical curves". They are maps from lines to higher dimensional Euclidean spaces.

We can consider another type of maps; "harmonic maps" from $\overline{\boldsymbol{I}}$ to a Riemannian manifold. In particular, when the target is an Euclidean space, we can consider harmonic embedding problem of graphs.

Definition 10 (cf. Kotani-Sunada) An edgewise C^1-map $f : \overline{I} \to \mathbb{R}^n$ is harmonic $\Leftrightarrow$ $f(I_{\{x,y\}})$ is a line joining $f(x)$ to $f(y)$ for $\{x, y\} \in E$, and

$$\sum_{y \in N(x)} C_{x,y} \frac{\partial f}{\partial r_{x,y}} = 0 \text{ for all } x \in V.$$

It is easy to see

$f : \overline{I} \to \mathbb{R}^n$ is harmonic $\Rightarrow$ $f(X)$ is a martingale on $\mathbb{R}^n$.

We obtained a simple result as follows.

Proposition 11 Assume an edgewise C^1-map $f : \overline{I} \to \mathbb{R}^n$ is a harmonic and isometric embedding. If X is conservative, $f(\overline{I})$ is not included in any non-degenerate cone.

3) Analogy to analogy.

Vojta considered analogous correspondence between Nevanlinna theory and Diophantine approximation. Then we may ask ;

Is there any correspondence between tropical Nevanlinna theory and Diophantine approx... ?