Quantitative fluctuation analysis of multiscale dynamical systems

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Outline

Problem formulation and motivation

The Stein-Malliavin method

Methodology and main results

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We consider a multiscale dynamical system of the form

$$\begin{cases} dX_t^\varepsilon &= c(X_t^\varepsilon, Y_t^\eta)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon, Y_t^\eta)dW_t^1 \\ dY_t^\eta &= \frac{1}{\eta} f(X_t^\varepsilon, Y_t^\eta)dt + \frac{1}{\sqrt{\eta}}\tau(X_t^\varepsilon, Y_t^\eta)dW_t^2 \end{cases}$$

X is called the *slow process* and Y the *fast process* as under the timescale $s = t/\eta$ for Y, one has

$$\begin{cases} dX_t^\varepsilon &= c(X_t^\varepsilon, Y_t^\eta)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon, Y_t^\eta)dW_t^1 \\ dY_s^\eta &= f(X_s^\varepsilon, Y_s^\eta)ds + \tau(X_s^\varepsilon, Y_s^\eta)dW_s^2 \end{cases}$$

Let $L_Y(x) = f(x,\cdot)\partial_y + \frac{1}{2}\tau^2(x,\cdot)\partial_y^2$ be the generator of Y.

Under an appropriate *recurrence condition* on f, $L_Y(x)$ admits an invariant measure μ.

Define the function $\bar{c}$ as

$$\bar{c}(x) = \int_{\mathbb{R}} c(x,y)\mu(dy)$$

and the *homogenization process* $\bar{X}$ as the solution to the ODE

$$\bar{X}_t = x + \int_0^t \bar{c}(\bar{X}_s)ds$$

We are interested in the fluctuation process

$$F_t^\varepsilon = \frac{X_t^\varepsilon - \bar{X}_t}{\sqrt{\varepsilon}},$$

where $\bar{X}$ is the homogenization process of X^ε as $\varepsilon, \eta \downarrow 0$.

Theorem (Spiliopoulos – 2014)

The process $\{F_t^\varepsilon : t \in [0,1]\}$ converges weakly in $\mathcal{C}([0,1])$ to a centered Ornstein-Uhlenbeck process.

We are interested in deriving *rates for the convergence* of F_t^{ε} to a Gaussian distribution.

For this purpose, we will use *second order Poincaré inequalities*.

The Stein-Malliavin method

From now on, let $\mathfrak{H} = L^2\left([0,1];\mathbb{R}^2\right)$.

A classical result in the Stein-Malliavin litterature states

Theorem (Nourdin, Peccati – 2009)

Let $F \in \mathbb{D}^{1,2}$ be a centered random variable. Then,

$$d_W(F, \mathcal{N}(0,1)) \leqslant \sqrt{\mathbb{E}\left(\left|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\right|^2\right)}$$
$$= \sqrt{\operatorname{Var}\left(\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\right)}$$

This result is very practical for *random variables F with known finite chaos decompositions*, but less tractable in general.

We know how to compute Malliavin derivatives of SDE's but not their image by L^{-1}.

The Gaussian *Poincaré inequality* states that, for any $F \in \mathbb{D}^{1,2}$,

$$\mathrm{Var}\,(F) \leqslant \mathbb{E}\left(\|DF\|_{\mathfrak{H}}^2\right)$$

This inequality can be applied to $F = \left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}}$ to get a bound *involving only the operator D*.

This is the object of the following theorem.

Theorem (Nourdin, Peccati, Reinert – 2009)
Let $F \in \mathbb{D}^{2,4}$ *with* $\mathbb{E}(F) = \mu$ *and* $\mathrm{Var}(F) = \sigma^2$. *Assume that* $N \sim \mathscr{N}(\mu, \sigma^2)$. *Then,*

$$d_W(F, N) \leqslant \frac{\sqrt{10}}{2\sigma^2} \mathbb{E}\left(\|DF\|_{\mathfrak{H}}^4\right)^{\frac{1}{4}} \mathbb{E}\left(\left\|D^2 F \otimes_1 D^2 F\right\|_{\mathfrak{H}^{\otimes 2}}^2\right)^{\frac{1}{4}}$$

Note that the above bound only involves D (and D^2).

A simple example: let $\{W_t : t \in [0,1]\}$ be a standard Brownian motion. Then,

$$DW_t = \mathbb{1}_{[0,t]} \quad \text{and} \quad D^2 W_t = 0,$$

so that

$$d_W(W_t, \mathcal{N}(0,t)) \leqslant \frac{\sqrt{10}}{2t} \sqrt{t}\, \mathbb{E}\left(\|0 \otimes_1 0\|^2_{L^2([0,1]^2)} \right)^{\frac{1}{4}} = 0$$

Remark

For normal convergence, first Malliavin derivatives converge to constants and second Malliavin derivatives converge to zero.

Methodology and main results

Recall that our fluctuation process is given by

$$F_t^\varepsilon = \frac{X_t^\varepsilon - \bar{X}_t}{\sqrt{\varepsilon}}$$

Let us introduce $\tilde{F}_t^\varepsilon$ defined by

$$\tilde{F}_t^\varepsilon = \frac{\sigma_t}{\sqrt{\operatorname{Var}\left(F_t^\varepsilon\right)}}\left[F_t^\varepsilon - \mathbb{E}\left(F_t^\varepsilon\right)\right],$$

where σ_t^2 is the variance of the limiting Gaussian distribution.

Let $N_t \sim \mathscr{N}\left(0, \sigma_t^2\right)$. We can then write

$$d_W\left(F_t^\varepsilon, N_t\right) \leqslant d_W\left(F_t^\varepsilon, \tilde{F}_t^\varepsilon\right) + d_W\left(\tilde{F}_t^\varepsilon, N_t\right)$$

We have

$$\begin{aligned} d_W\left(F_t^\varepsilon, \tilde{F}_t^\varepsilon\right) &\leqslant \mathbb{E}\left(\left|F_t^\varepsilon - \tilde{F}_t^\varepsilon\right|\right) \\ &\leqslant \mathbb{E}\left(\left|F_t^\varepsilon\right|\right)\left|1 - \frac{\sigma_t}{\sqrt{\operatorname{Var}\left(F_t^\varepsilon\right)}}\right| + \frac{\sigma_t}{\sqrt{\operatorname{Var}\left(F_t^\varepsilon\right)}}\left|\mathbb{E}\left(F_t^\varepsilon\right)\right| \end{aligned}$$

Using the "Poincaré" bound for $d_W\left(\tilde{F}_t^{\varepsilon}, N_t\right)$, we get in total

$$d_W\left(F_t^{\varepsilon}, N_t\right) \leqslant \mathbb{E}\left(\left|F_t^{\varepsilon}\right|\right)\left|1-\frac{\sigma_t}{\sqrt{\operatorname{Var}\left(F_t^{\varepsilon}\right)}}\right|+\frac{\sigma_t}{\sqrt{\operatorname{Var}\left(F_t^{\varepsilon}\right)}}\left|\mathbb{E}\left(F_t^{\varepsilon}\right)\right| + \frac{\sqrt{10}}{2\varepsilon \operatorname{Var}\left(F_t^{\varepsilon}\right)} \mathbb{E}\left(\left\|D X_t^{\varepsilon}\right\|_{\mathfrak{H}}^4\right)^{\frac{1}{4}} \mathbb{E}\left(\left\|D^2 X_t^{\varepsilon} \otimes_1 D^2 X_t^{\varepsilon}\right\|_{\mathfrak{H}^{\otimes 2}}^2\right)^{\frac{1}{4}}$$

Step 1: Convergence of the first two moments

We need to deal with $|\mathbb{E}(F_t^\varepsilon)|$ and $\left|\mathrm{Var}(F_t^\varepsilon) - \sigma_t^2\right|$. We have

$$F_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\left(\int_0^t c(X_s^\varepsilon, Y_s^\eta)ds - \int_0^t \bar{c}(\bar{X}_s)ds\right) + \int_0^t \sigma(X_s^\varepsilon, Y_s^\eta)dW_s^1$$

As is, the quantity $c(X_s^\varepsilon, Y_s^\eta) - \bar{c}(\bar{X}_s)$ is hard to deal with.

We can write

$$c(X_s^\varepsilon, Y_s^\eta) - \bar{c}(\bar{X}_s) = c(X_s^\varepsilon, Y_s^\eta) - \bar{c}(X_s^\varepsilon) + \bar{c}(X_s^\varepsilon) - \bar{c}(\bar{X}_s)$$

Taylor's theorem implies that

$$\bar{c}(X_s^\varepsilon) - \bar{c}(\bar{X}_s) = \partial_x \bar{c}(\bar{X}_s))(X_s^\varepsilon - \bar{X}_s) + \Lambda[\bar{c}](X_s^\varepsilon, \bar{X}_s)$$

so that

$$\frac{\bar{c}(X_s^\varepsilon) - \bar{c}(\bar{X}_s)}{\sqrt{\varepsilon}} = \partial_x \bar{c}(\bar{X}_s))F_s^\varepsilon + \frac{\Lambda[\bar{c}](X_s^\varepsilon, \bar{X}_s)}{\sqrt{\varepsilon}}$$

Consider the Poisson partial differential equation

$$L_Y \Phi(x, y) = c(x, y) - \bar{c}(x)$$

with boundary condition

$$\int_{\mathbb{R}} \Phi(\cdot, y)\mu(dy) = 0$$

Then,

$$\begin{aligned} c(X_s^\varepsilon, Y_s^\eta) - \bar{c}(X_s^\varepsilon) &= L_Y \Phi(X_s^\varepsilon, Y_s^\eta) \\ &= f(X_s^\varepsilon, Y_s^\eta)\partial_y \Phi(X_s^\varepsilon, Y_s^\eta) + \frac{1}{2}\tau^2(X_s^\varepsilon, Y_s^\eta)\partial_y^2 \Phi(X_s^\varepsilon, Y_s^\eta) \end{aligned}$$

In total, we have the new representation

$$
\begin{aligned}
F_t^\varepsilon = & \int_0^t \partial_x \bar{c}(\bar{X}_s) F_s^\varepsilon ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t \Lambda[\bar{c}](X_s^\varepsilon, \bar{X}_s) ds \\
& + \frac{\eta}{\sqrt{\varepsilon}} \left(\Phi(X_t^\varepsilon, Y_t^\eta) - \Phi(X_0^\varepsilon, Y_0^\eta) \right) \\
& - \frac{\eta}{\sqrt{\varepsilon}} \int_0^t \Big(c(X_s^\varepsilon, Y_s^\eta) \partial_x \Phi(X_s^\varepsilon, Y_s^\eta) + \frac{\varepsilon}{2} \sigma(X_s^\varepsilon, Y_s^\eta)^2 \partial_x^2 \Phi(X_s^\varepsilon, Y_s^\eta) \Big) ds \\
& + \int_0^t \left(1 - \eta \partial_x \Phi(X_s^\varepsilon, Y_s^\eta)\right) \sigma(X_s^\varepsilon, Y_s^\eta) dW_s^1 \\
& - \sqrt{\frac{\eta}{\varepsilon}} \int_0^t \partial_{x_2} \Phi(X_s^\varepsilon, Y_s^\eta) \tau(X_s^\varepsilon, Y_s^\eta) dW_s^2
\end{aligned}
$$

Step 2: Convergence of the Malliavin derivatives

We need to estimate $\mathbb{E}\left(\|DX_t^\varepsilon\|_{\mathfrak{H}}^4\right)$ and $\mathbb{E}\left(\left\|D^2X_t^\varepsilon \otimes_1 D^2X_t^\varepsilon\right\|_{\mathfrak{H}^{\otimes 2}}^2\right)$ in terms of ε and η.

We will illustrate the main ideas for the first derivatives.

We have

$$D_r^{W^1} X_t^\varepsilon = \sqrt{\varepsilon}\sigma\left(X_r^\varepsilon, Y_r^\eta\right) + \int_r^t \left[\partial_1 c\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} X_s^\varepsilon + \partial_2 c\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} Y_s^\eta\right] ds + \sqrt{\varepsilon}\int_r^t \left[\partial_1 \sigma\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} X_s^\varepsilon + \partial_2 \sigma\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} Y_s^\eta\right] dW_s^1$$

$$D_r^{W^1} Y_t^\eta = \frac{1}{\eta}\int_r^t \left[\partial_1 f\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} X_s^\varepsilon + \partial_2 f\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} Y_s^\eta\right] ds + \frac{1}{\sqrt{\eta}}\int_r^t \left[\partial_1 \tau\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} X_s^\varepsilon + \partial_2 \tau\left(X_s^\varepsilon, Y_s^\eta\right) D_r^{W^1} Y_s^\eta\right] dW_s^2$$

We can write

$$\mathbb{E}\left(\sup_{r\leqslant s\leqslant t}\left|D_r^{W^1}X_s^{\varepsilon}\right|^{2p}\right)$$
$$\leqslant C\left(\varepsilon^p + (1+\varepsilon^p)\left[\int_r^t \mathbb{E}\left(\sup_{r\leqslant u\leqslant s}\left|D_r^{W^1}X_u^{\varepsilon}\right|^{2p}\right)ds\right.\right.$$
$$\left.\left. + \mathbb{E}\left(\int_r^t \left|D_r^{W^1}Y_s^{\eta}\right|^{2p} ds\right)\right]\right)$$

The Itô formula implies

$$\mathbb{E}\left(\left|D_r^{W^1}Y_t^\eta\right|^{2p}\right) = \frac{2p}{\eta}\mathbb{E}\Bigg(\int_r^t \left(D_r^{W^1}Y_s^\eta\right)^{2p-1}\Big[\partial_1 f\left(X_s^\varepsilon, Y_s^\eta\right)D_r^{W^1}X_s^\varepsilon + \partial_2 f\left(X_s^\varepsilon, Y_s^\eta\right)D_r^{W^1}Y_s^\eta\Big]ds\Bigg)$$
$$+ \frac{2p(2p-1)}{\eta}\mathbb{E}\Bigg(\int_r^t \left(D_r^{W^1}Y_s^\eta\right)^{2p-2}\Big[\partial_1 \tau\left(X_s^\varepsilon, Y_s^\eta\right)D_r^{W^1}X_s^\varepsilon + \partial_2 \tau\left(X_s^\varepsilon, Y_s^\eta\right)D_r^{W^1}Y_s^\eta\Big]^2 ds\Bigg)$$

Applying Young's inequality for products yields

$$\begin{aligned}\mathbb{E}\left(\left|D_r^{W^1} Y_t^\eta\right|^{2p}\right) \leqslant \frac{1}{\eta}\mathbb{E}\Bigg(\int_r^t \Big[&\left|\partial_1 f\left(X_s^\varepsilon, Y_s^\eta\right)\right| \\ &+ 2(2p-1)\left|\partial_1 \tau\left(X_s^\varepsilon, Y_s^\eta\right)\right|^2 \Big] \left|D_r^{W^1} X_s^\varepsilon\right|^{2p} ds \Bigg) \\ + \frac{1}{\eta}\mathbb{E}\Bigg(\int_r^t \Big[&2p\partial_2 f\left(X_s^\varepsilon, Y_s^\eta\right) + (2p-1)\left|\partial_1 f\left(X_s^\varepsilon, Y_s^\eta\right)\right| \\ &+ (2p-1)(2p-2)\left|\partial_1 \tau\left(X_s^\varepsilon, Y_s^\eta\right)\right|^2 \\ &+ 2p(2p-1)\left|\partial_2 \tau\left(X_s^\varepsilon, Y_s^\eta\right)\right|^2 \Big] \left|D_r^{W^1} Y_s^\eta\right|^{2p} ds \Bigg)\end{aligned}$$

The recurrence relation states

$$\sup_{x,y\in\mathbb{R}} \left\{ \left[4\partial_2 f + 3\left|\partial_1 f\right| + 6\left|\partial_1 \tau\right|^2 + 12\left|\partial_2 \tau\right|^2 \right](x,y) \right\} \leqslant -K < 0$$

Essentially, it requires $\partial_2 f$ to be sufficiently negative.

We also have the following boundedness assumption

$$\sup_{x,y\in\mathbb{R}} \left\{ \left[|\partial_1 f| + 6|\partial_1 \tau|^2 \right](x,y) \right\} < M$$

We hence get

$$\mathbb{E}\left(\left|D_r^{W^1} Y_t^\eta\right|^{2p}\right) \leqslant \frac{M}{\eta}\mathbb{E}\left(\int_r^t \left|D_r^{W^1} X_s^\varepsilon\right|^{2p} ds\right) - \frac{K}{\eta}\mathbb{E}\left(\int_r^t \left|D_r^{W^1} Y_s^\eta\right|^{2p} ds\right),$$

so that

$$\mathbb{E}\left(\int_r^t \left|D_r^{W^1} Y_s^\eta\right|^{2p} ds\right) \leqslant \frac{M}{K}\mathbb{E}\left(\int_r^t \left|D_r^{W^1} X_s^\varepsilon\right|^{2p} ds\right)$$

This yields

$$\mathbb{E}\left(\sup_{r\leqslant s\leqslant t}\left|D_r^{W^1}X_s^\varepsilon\right|^{2p}\right)$$
$$\leqslant C\left(\varepsilon^p+(1+\varepsilon^p)\int_r^t\mathbb{E}\left(\sup_{r\leqslant u\leqslant s}\left|D_r^{W^1}X_u^\varepsilon\right|^{2p}\right)ds\right)$$

By Grönwall's lemma, this implies

$$\mathbb{E}\left(\sup_{r\leqslant s\leqslant t}\left|D_r^{W^1}X_s^\varepsilon\right|^{2p}\right)\leqslant C\varepsilon^p$$

Assuming that $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon \downarrow 0} \sqrt{\frac{\varepsilon}{\eta}} = \gamma \in (0, \infty],$$

we obtain the following quantitative result.

Theorem (B., Spiliopoulos – 2023)

Let the above assumptions and notation prevail. Then, for any $\zeta > 0$,

$$\sup_{t \in (0,T]} d_W\left(F_t^{\varepsilon}, N_t\right) \lesssim \sqrt{\sqrt{\varepsilon} + \sqrt{\eta} + \left(\frac{\eta}{\varepsilon} - \frac{1}{\gamma^2}\right) + \frac{\eta^{1-\zeta}}{\sqrt{\varepsilon}} + \frac{\sqrt{\eta^3}}{\varepsilon} + \frac{\sqrt{\eta^5}}{\varepsilon^2}}$$

Thank you!