Global solutions to the stochastic reaction-diffusion equation with superlinear forcing and superlinear multiplicative noise

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Bounded spatial domain \( x \in D \subset \mathbb{R}^d \).

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\begin{cases}
\frac{\partial u}{\partial t}(t, x) = A u(t, x) + f(u(t, x)) + \sigma(u(t, x)) \dot{w}(t, x) & t > 0, \ x \in D, \\
u(t, x) = 0, \ x \in \partial D
\end{cases}
\]

- Second-order elliptic differential operator
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  A = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right).
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Stochastic reaction-diffusion equation

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- Nonlinear reaction term $f(u(t, x))$
- Multiplicative noise term $\sigma(u(t, x)) \dot{w}$.
The solution to the non-stochastic heat equation \( \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u \), \( u(0, x) = u_0(x) \) can be written as a convolution with a heat kernel

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    u(t, x) = \int_D K(t, x, y)u_0(y)dy. \tag{1}
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If the domain is the whole space then \( K(t, x, y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \).
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Duhamel’s principle gives the solution to the semilinear heat equation $\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + f(u)$

$$u(t, x) = \int_D K(t, x, y)u_0(y)dy + \int_0^t \int_D K(t-s, x, y)f(u(s, y))dyds.$$
This convolution is a semigroup $S(t) : C(D) \rightarrow C(D)$

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Formally, solution should be given by

$$u(t) = S(t)u_0 + \int_0^t S(t - s)f(u(s))ds + \int_0^t S(t - s)\sigma(u(s))dw(s).$$
Existence of global solutions

- Under reasonable assumptions on $A$ and $\dot{w}$, the solutions are function-valued.

- If $f$ and $\sigma$ are globally Lipschitz continuous, there exists a unique solution and solutions exist for all $t > 0$. (Da Prato, Zabczyk)

- If $\sigma$ is locally Lipschitz continuous with linear growth and $f$ is locally Lipschitz continuous and features dissipative behaviors $\lim_{|x| \to \infty} f(x) = -\infty$, global solutions exist. (Brzezniak Peszat 1999, Iwata 1999, Cerrai 2003, Da Prato Röckner 2002, Marinelli Röckner 2010, Röckner Liu 2010)

- Mueller and collaborators (1991, with Sowers 1993, 1998, 2000) showed that in the space-time white noise case with $A = \partial^2_{xx}$, $f \equiv 0$ and $\sigma(u) = |u|^{\gamma}$, solutions never explode if $\gamma < \frac{3}{2}$ and explode with positive probability if $\gamma > \frac{3}{2}$.
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- Dissipative forces push solutions away from \( \pm \infty \), counteracting expansion due to noise.
- I outline some recent results about sufficient conditions in the accretive and dissipative settings that guarantee that mild solutions to the SRDE do not explode.
Technical Assumptions on $A$ and $\dot{w}$

- Eigenvalues of $A$, $\{e_k(x)\}$ complete orthonormal basis of $L^2(D)$.
  \[ A e_k = -\alpha_k e_k, \quad 0 \leq \alpha_k \leq \alpha_{k+1} \].

Formal definition of the noise: Sequence $\lambda_j \geq 0$

\[ \dot{w}(t, x) = \infty \sum_{j=1}^{\infty} \lambda_j e_j(x) \, d\beta_j(t), \quad \beta_j(t) \text{ i.i.d. one-dimensional B.M.} \]

Condition (Cerrai 2003) - There exist $\theta > 0$, $\rho \in [2, +\infty)$

\[ \sum_{k=1}^{\infty} \alpha_k - \theta k |e_k|^2_{L^\infty} < +\infty, \quad \sum_{j=1}^{\infty} \lambda_j \rho |e_j|^2_{L^\infty} < +\infty \text{ or } \sup_j \lambda_j < +\infty \quad (\rho = +\infty) \]

\[ \eta := \theta (\rho - 2), \quad \rho < 1 \quad (\eta := \theta \text{ if } \rho = +\infty) \]
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- Space Hölder continuity $\approx (1 - \eta)$. Time Hölder continuity $\approx \frac{1-\eta}{2}$. 

Salins (BU) Superlinear SPDE June 27, 2023 7 / 28
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Example

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- If $d = 1$ we can take $\lambda_{j} \equiv 1$ (space-time white noise), $\rho = \infty$ $\theta \in (1/2, 1)$, $\eta = \theta$ arbitrarily close to $1/2$. 
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- If $d > 1$, space-time white noise not allowed. On rectangular domains we need $\theta > \frac{d}{2}$, $\rho < \frac{2d}{d-2}$. 
Superlinear $\sigma$ when $f \equiv 0$


Heat equation on one spatial dimension. Space-time white noise.

Intuition: The $t^{-1/2}$ in the heat kernel $G(t, x, y) \approx e^{-|x-y|^2 / 2t \sqrt{2\pi t}}$ allows for extra growth.

Generalization to higher spatial dimensions and other more general conditions in dissertation by Bezdek.
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Polynomially dissipative $f$ – SDE example

- SDE on $\mathbb{R}^d$

$$dX(t) = -X(t)|X(t)|^{\beta-1}dt + (1 + |X(t)|)^\gamma dW(t)$$
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$$\mathbb{E}|X(t)|^2 = \mathbb{E}|X(0)|^2 + \mathbb{E} \int_0^t \left(-2|X(s)|^{\beta+1} + (1 + |X(s)|^{\gamma})^2\right) ds.$$
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- Similar conditions identified by Liu and Röckner (2010) for monotone SPDEs

\[ 2V^* \langle A(t, v), v \rangle_V + \|B(t, v)\|_V^2 + \theta \|v\|_V^\alpha \leq C + K \|v\|_H^2. \]
Goal: identify sufficient conditions on dissipativity strength of $f$ and growth rate of $\sigma$ to guarantee that mild solutions to SRDE never explode with non-trace-class noise.
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Assume $f(u)\text{sign}(u) \leq -\mu |u|^{\beta}$ for $|u| > c_0$,
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Note: Trace-class noise means $\eta = 0$ and this coincides with the Itô formula condition $\gamma < \beta + 1/2$. When $\eta$ not trace-class, Itô formula arguments are not available. As long as $\beta > 1$, $\gamma$ can grow superlinearly.
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Idea of the proof

- Set up a sequence of stopping times

\[ \tau_0 = \inf \{ t \geq 0 : |u(t)|_{L^\infty(D)} = 3^n c_0 \text{ for some } n \in \{1, 2, 3, \ldots\} \}, \]
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- If \(|u(\tau_n)|_{L^\infty} \geq 3^2 c_0\),
  \[ \tau_{n+1} = \inf \left\{ t \geq \tau_n : \left| u(t) \right|_{L^\infty(D)} = 3 \left| u(\tau_k) \right|_{L^\infty(D)} \text{ or } \frac{1}{3} \left| u(\tau_n) \right|_{L^\infty(D)} \right\} \]
Idea of the proof

- Set up a sequence of stopping times

\[ \tau_0 = \inf\{ t \geq 0 : |u(t)|_{L^\infty(D)} = 3^n c_0 \text{ for some } n \in \{1, 2, 3, \ldots\} \}, \]

- If \( |u(\tau_n)|_{L^\infty} \geq 3^2 c_0 \),

\[ \tau_{n+1} = \inf\left\{ t \geq \tau_n : |u(t)|_{L^\infty(D)} = 3|u(\tau_k)|_{L^\infty(D)} \text{ or } \frac{1}{3}|u(\tau_n)|_{L^\infty(D)} \right\} \]

- If \( |u(\tau_n)|_{L^\infty(D)} = 3c_0 \),

\[ \tau_{n+1} = \inf\{ t \geq \tau_n : |u(t)|_{L^\infty(D)} = 3|u(\tau_k)|_{L^\infty(D)} \}. \]
Idea of the proof

- There exist $C > 0$, $q > 1$ such that for any $k \in \mathbb{N}$, $\varepsilon > 0$,

$$
P(|u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty} \text{ and } \tau_{k+1} - \tau_k < \varepsilon) \leq C\varepsilon^q.
$$
Idea of the proof

- There exist $C > 0$, $q > 1$ such that for any $k \in \mathbb{N}$, $\varepsilon > 0$,

$$\mathbb{P} \left( |u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty} \text{ and } \tau_{k+1} - \tau_k < \varepsilon \right) \leq C\varepsilon^q.$$  

- By Borel-Cantelli setting $\varepsilon = \frac{1}{k}$,

$$|u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty} \text{ and } \tau_{k+1} - \tau_k < \frac{1}{k} \text{ a finite number of times.}$$
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$$

- Eventually, $|u(\tau_k)|_{L^\infty}$ decreases or it takes more than $1/k$ time to triple. If it explodes it has more up steps than down steps.
Idea of the proof

- There exist $C > 0$, $q > 1$ such that for any $k \in \mathbb{N}$, $\varepsilon > 0$,

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  a finite number of times.

- Eventually, $|u(\tau_k)|_{L^\infty}$ decreases or it takes more than $1/k$ time to triple. If it explodes it has more up steps than down steps.

- This is enough to prove that $\sum_k (\tau_{k+1} - \tau_k) = +\infty$. Cannot explode in finite time.
Idea of the proof

- If $\beta > 1$, the deterministic ODE $\frac{dv}{dt} = -|v(t)|^{\beta} \text{sign}(v(t))$ has solution $|v(t)| = |v(0)| - (\beta - 1) + Ct^{\beta - 1} \leq \max\{|v(0)|, Ct^{\beta - 1}\}$.
If $\beta > 1$, the deterministic ODE $\frac{dv}{dt} = -|v(t)|^\beta \text{sign}(v(t))$ has solution $|v(t)| = (|v(0)|^{-(\beta-1)} + Ct)^{-\frac{1}{\beta-1}}$. 
Idea of the proof

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- The mild solution to the SPDE

$$u(t + \tau_k) = S(t)u(\tau_k) + \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)f(u(s))ds + \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)\sigma(u(s))dW(s)$$
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The mild solution to the SPDE

$$u(t + \tau_k) = S(t)u(\tau_k) + \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)f(u(s))ds + \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)\sigma(u(s))dW(s)$$

If $Z_k(t) := \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)\sigma(u(s))dW(s)$ satisfies $|Z_k(t)|_{L^\infty} \leq \frac{1}{3}|u(t + \tau_k)|_{L^\infty}$, for $t \in [0, \tau_k+1 - \tau_k]$, then

$$|u(t)|_{L^\infty} \leq \frac{3}{2} \left(|u(\tau_k)|_{L^\infty}^{-(\beta-1)} + Ct\right)^{-\frac{1}{\beta-1}}.$$
Idea of the proof

- Moment bounds (Cerrai 2003) – For $p > 1$ large enough, and small numbers $\alpha, \gamma \in (0, (1 - \eta)/2)$ such that $(\alpha - \frac{\zeta}{2})p > 1$ and

$$
\mathbb{E} \sup_{t \in [0, T]} \sup_{x \in D} |Z_k(t, x)|^p \leq C T^{p(\alpha - \frac{\zeta}{2}) - 1} \int_0^T \mathbb{E} \left( \int_0^t (t - s)^{-2\alpha - \eta} |\sigma(u(s + \tau_k))|_{L^\infty}^2 ds \right)^{\frac{p}{2}} dt.
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Idea of the proof

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$$

- $u(t + \tau_k)$ will decay like $t^{-\frac{1}{\beta - 1}}$ when $|Z_k(t)|_{L^\infty} \leq \frac{1}{3} |u(t + \tau_k)|_{L^\infty}$ so inner integral is bounded by

$$
C \int_0^t (t - s)^{-2\alpha - \eta} |u(\tau_k)|^{2}_{L^\infty} s^{-\frac{2(\gamma - 1)}{\beta - 1}} ds.
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$$C \int_0^t (t - s)^{-2\alpha - \eta} |u(\tau_k)|_{L^\infty}^2 s^{-\frac{2(\gamma - 1)}{\beta - 1}} ds.$$

- When $2\alpha + \eta + \frac{2(\gamma - 1)}{\beta - 1} < 1$, this inner integral is bounded as $t \downarrow 0$. (Beta function)
Idea of proof

- $\mathbb{P} \left( |u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty}, \tau_{k+1} - \tau_k < \varepsilon \left| u(\tau_k) \right|_{L^\infty} = c_0 3^n \right) \leq \mathbb{P} \left( \sup_{t \in [0, \varepsilon \wedge (\tau_{k+1} - \tau_k)]} |Z_k(t)|_{L^\infty} \geq \frac{1}{9} |u(\tau_k)|_{L^\infty} \left| u(\tau_k) \right|_{L^\infty} = c_0 3^n \right)$. 

By Chebyshev's inequality, $\leq C\varepsilon (\alpha - \zeta^2)^{\frac{3}{3-n}}$. 

Can be chosen so that $q = \alpha - \zeta^2 > 1$. 

Setting $\varepsilon = 1/k$ and using the Borel-Centelli Lemma tells us that eventually it takes more than $1/k$ time for the $L^\infty$ norm to triple. If it were to explode, then it would need more up steps than down steps. This is enough to guarantee that $\lim_{k \to \infty} \tau_k = +\infty$. 

Idea of proof

- \( \mathbb{P}\left(|u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty}, \tau_{k+1} - \tau_k < \varepsilon \left| u(\tau_k)|_{L^\infty} = c_03^n \right) \right) \leq \mathbb{P}\left( \sup_{t \in [0, \varepsilon \wedge (\tau_{k+1} - \tau_k)]} |Z_k(t)|_{L^\infty} \geq \frac{1}{9} |u(\tau_k)|_{L^\infty} \left| u(\tau_k)|_{L^\infty} = c_03^n \right) \right) \cdot \)

- By Chebyshev,

\[
\leq C \varepsilon^{(\alpha - \frac{\xi}{2})p} \frac{3^{np}}{3^{np}} \leq C \varepsilon^{(\alpha - \frac{\xi}{2})p},
\]
Idea of proof

- \( \mathbb{P} \left( \left| u(\tau_{k+1}) \right|_{L^\infty} = 3 \left| u(\tau_k) \right|_{L^\infty}, \tau_{k+1} - \tau_k < \varepsilon \left| u(\tau_k) \right|_{L^\infty} = c_0 3^n \right) \leq \mathbb{P} \left( \sup_{t \in [0, \varepsilon \wedge (\tau_{k+1} - \tau_k)]} \left| Z_k(t) \right|_{L^\infty} \geq \frac{1}{9} \left| u(\tau_k) \right|_{L^\infty} \right) \).

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- Setting \( \varepsilon = 1/k \) and using Borel-Centelli Lemma tell us that eventually it takes more than \( 1/k \) time for the \( L^\infty \) norm to triple.
Idea of proof

- \( \mathbb{P} \left( \left| u(\tau_{k+1}) \right|_{L^\infty} = 3 \left| u(\tau_k) \right|_{L^\infty}, \tau_{k+1} - \tau_k < \varepsilon \left| u(\tau_k) \right|_{L^\infty} = c_0 3^n \right) \leq \mathbb{P} \left( \sup_{t \in [0, \varepsilon \wedge (\tau_{k+1} - \tau_k)]} |Z_k(t)|_{L^\infty} \geq \frac{1}{9} \left| u(\tau_k) \right|_{L^\infty} \right) \).
- By Chebyshev,
  \[ \leq C \varepsilon^\left( \alpha - \frac{\zeta}{2} \right) p \frac{3^{np}}{3^{np}} \leq C \varepsilon^\left( \alpha - \frac{\zeta}{2} \right) p, \]

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- Setting \( \varepsilon = 1/k \) and using Borel-Centelli Lemma tell us that eventually it takes more than \( 1/k \) time for the \( L^\infty \) norm to triple.
- If it were to explode, then it would need more up steps than down steps. This is enough to guarantee that \( \lim \tau_k = +\infty. \)
Theorem (S. 2022 – same theorem)

Assume \( f(u)\text{sign}(u) \leq -\mu |u|^{\beta} \) for \( |u| > c_0 \), \( |\sigma(u)| \leq C(1 + |u|^{\gamma}) \), with \( \gamma < 1 + \frac{(1-\eta)(\beta-1)}{2} \). Then mild solutions can never explode.
Example

Theorem (S. 2022 – same theorem)

Assume \( f(u) \text{sign}(u) \leq -\mu |u|^\beta \) for \(|u| > c_0\), \(|\sigma(u)| \leq C(1 + |u|^\gamma)\), with \( \gamma < 1 + \frac{(1-\eta)(\beta-1)}{2} \). Then mild solutions can never explode.

- Consider the non-linear heat equation on a one-dimensional interval with space-time white noise

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) - |u(t, x)|^\beta \text{sign}(u(t, x)) + (1 + |u(t, x)|)^\gamma \dot{w}(t, x).
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Assume $f(u)\text{sign}(u) \leq -\mu |u|^{\beta} \text{ for } |u| > c_0$, $|\sigma(u)| \leq C(1 + |u|^{\gamma})$, with $\gamma < 1 + \frac{(1-\eta)(\beta-1)}{2}$. Then mild solutions can never explode.

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- $\eta$ can be chosen arbitrarily close to $\frac{1}{2}$ so if $\gamma < 1 + \frac{\beta-1}{4}$ then solutions cannot explode in finite time.
Theorem (S. 2022 – same theorem)

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- Consider the non-linear heat equation on a one-dimensional interval with space-time white noise

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- If $\beta \leq 3$, the Mueller results give a stronger result, so $\gamma$ can be any value $\gamma < \frac{3}{2}$. 
Theorem (S. 2022 – same theorem)

Assume \( f(u)\) sign\( (u) \leq -\mu |u|^\beta \) for \( |u| > c_0, \) \( |\sigma(u)| \leq C(1 + |u|^\gamma) \), with \( \gamma < 1 + \frac{(1-\eta)(\beta-1)}{2} \). Then mild solutions can never explode.

- Consider the non-linear heat equation on a one-dimensional interval with space-time white noise

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\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) - |u(t, x)|^\beta \text{sign}(u(t, x)) + (1 + |u(t, x)|)^\gamma \dot{w}(t, x).
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- \( \eta \) can be chosen arbitrarily close to \( \frac{1}{2} \) so if \( \gamma < 1 + \frac{\beta-1}{4} \) then solutions cannot explode in finite time.
- If \( \beta \leq 3 \), the Mueller results give a stronger result, so \( \gamma \) can be any value \( \gamma < \frac{3}{2} \).
  - The semigroup causes \( t^{-\frac{1}{2}} \) decay but the nonlinear term causes \( t^{-\frac{1}{\beta-1}} \) decay.
Now we see the effect of superlinear accretive forcing

\[ \lim_{|x| \to \infty} f(x) = \pm \infty. \]

Such a force pushes solutions toward \( \pm \infty \).

Does it cause explosion in finite time?

Osgood condition for ODE – Assume \( f \geq 0 \)

\[ \frac{dv}{dt} = f(v(t)), \quad v(0) = c > 0, \]

Explodes in finite time if and only if

\[ \int_{c}^{\infty} f(x) \, dx < \infty. \]
Now we see the effect of superlinear accretive $f$

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Osgood condition for ODE – Assume $f \geq 0$

$$\frac{dv}{dt} = f(v(t)), \ v(0) = c > 0,$$

Explodes in finite time if and only if $\int_c^\infty \frac{1}{f(x)} \, dx < +\infty$. 
Examples of exploding/non-exploding ODEs

\[ \frac{dv}{dt} = f(v(t)) \text{ explodes iff } \int_c^\infty \frac{1}{f(x)} \, dx < +\infty. \]
Examples of exploding/non-exploding ODEs

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\[f(u) = u.\]
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- \( f(u) = u^{1+\varepsilon} \) explodes in finite time. Solution: \( v(t) = (C - \varepsilon t)^{-\frac{1}{\varepsilon}} \).
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Superlinear accretive ODE

Examples of exploding/non-exploding ODEs

\[ \frac{dv}{dt} = f(v(t)) \text{ explodes iff } \int_{c}^{\infty} \frac{1}{f(x)} dx < +\infty. \]

- \( f(u) = u. \) Solution: \( v(t) = e^t. \)
- \( f(u) = u^{1+\epsilon} \) explodes in finite time. Solution: \( v(t) = (C - \epsilon t)^{-\frac{1}{\epsilon}}. \)
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Examples of exploding/non-exploding ODEs

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For deterministic PDEs, Osgood condition does not fully characterize explosion (Fujita 1966). \( \frac{\partial u}{\partial t} = \Delta u + |u|^p \) if \( p > 1 + \frac{2}{d} \).
For deterministic PDEs, Osgood condition does not fully characterize explosion (Fujita 1966). \( \frac{\partial u}{\partial t} = \Delta u + |u|^p \) if \( p > 1 + \frac{2}{d} \).

Bonder and Groisman (2009) showed that in the case of additive noise \( \sigma(u) \equiv 1 > 0 \), if \( \int_{c}^{\infty} \frac{1}{f(x)} \, dx < \infty \) for some \( c > 0 \), then the SRDE explodes in finite time with probability one.
Osgood condition for SRDE \( \frac{\partial u}{\partial t} = Au + f(u) + \sigma(u)\dot{W} \)

- For deterministic PDEs, Osgood condition does not fully characterize explosion (Fujita 1966). \( \frac{\partial u}{\partial t} = \Delta u + |u|^p \) if \( p > 1 + \frac{2}{d} \).
- Bonder and Groisman (2009) showed that in the case of additive noise \( \sigma(u) \equiv 1 > 0 \), if \( \int_c^\infty \frac{1}{f(x)}dx < \infty \) for some \( c > 0 \), then the SRDE explodes in finite time with probability one.
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- Foondun and Nualart (2021) prove that Osgood condition is a characterization of explosion when $\sigma$ is constant.
  - Explosion in finite time if $\int_c^\infty \frac{1}{f(x)} dx < +\infty$ for some $c > 0$.
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- Dalang, Khoshnevisan, and Zhang (2019) showed that $\sigma$ can be superlinear too. Studied the space-time white noise on a bounded one-dimensional spatial domain
  - Assume $|f(u)| \leq C(1 + |u| \log |u|)$, $\sigma \in o(|u|(\log |u|)^{1/4})$.
  - Then solutions never explode.
Theorem (S. 2022)

Assume \( f \) and \( \sigma \) are locally Lipschitz continuous functions. Assume that there exists a positive, increasing function \( h : [0, +\infty) \to [0, +\infty) \) such that

\[
\int_0^\infty \frac{1}{h(u)} du = \infty \quad \text{and} \quad |f(u)| \leq h(|u|)
\]

and there exists \( \gamma \in \left(0, \frac{1-\eta}{2}\right) \) such that

\[
|\sigma(u)| \leq |u|^{1-\gamma}(h(|u|))^{\gamma} \quad \text{for all } |u| > 1.
\]

Assume the initial data is bounded. Then there exists a unique global solution.
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If \( h(u) = u \log(u) \log \log(u) \) then \( \sigma(u) \leq u \left( \log u \log \log u \right)^{\gamma} \).
Define the cutoff versions of \( f \) and \( \sigma \)

\[
    f_n(x) = \begin{cases} 
        f(x) & \text{if } x \in [-3^n, 3^n] \\
        f(3^n) & \text{if } x > 3^n \\
        f(-3^n) & \text{if } x < -3^n 
    \end{cases}
\]

By Cerrai (2003), there is a unique solution to

\[
    u_n(t) = S(t) u(0) + \int_0^t S(t-s) f_n(u_n(s)) \, ds + \int_0^t S(t-s) \sigma_n(u_n(s)) \, dw(s).
\]

Define stopping time \( \tau_n = \inf \{ t > 0 : \sup_{x \in D} |u_n(t, x)| > 3^n \} \).

Define local mild solution \( u(t, x) = u_n(t, x) \) for all \( t < \tau_n \). Solution EXPLODES in finite time if \( \sup_n \tau_n < +\infty \) and solution is global in time if \( \sup_n \tau_n = +\infty \).
Local mild solution and explosion

- Define the cutoff versions of $f$ and $\sigma$

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Salins (BU)  Superlinear SPDE  June 27, 2023  22 / 28
Local mild solution and explosion

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Moment bounds on the stochastic convolution

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Moment bounds on the stochastic convolution

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- For any $\zeta \in (0, 1 - \eta)$, $p > \max \left\{ \frac{2}{1 - \eta - \zeta}, \frac{d}{\zeta} \right\}$, there exists $C = C(\zeta, p)$ such that for any adapted random field that is almost surely bounded

$$\mathbb{P} \left( \sup_{s \in [0,t]} \sup_{x \in D} |\Phi(t, x)| \leq M \right) = 1.$$ 

$$\mathbb{E} \sup_{t \in [0,\varepsilon]} \left| \int_0^t S(t - s)\Phi(s)dw(s) \right|_{L^\infty}^p \leq C M^p \varepsilon^{\frac{p(1 - \eta - \zeta)}{2}}. \quad (2)$$
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$$

• One interpretation is that time Hölder continuity is slightly worse than $\frac{1 - \eta}{2} -$ same number that describes how superlinear $\sigma$ can be.
Idea of the proof

Consider the ODE \( \frac{dv}{dt} = f(v(t)) \) where \( |f(v)| \leq h(|v|) \) and \( \int_0^\infty \frac{1}{h(v)} dv = +\infty \) and \( h \) increasing.
Idea of the proof

- Consider the ODE $\frac{dv}{dt} = f(v(t))$ where $|f(v)| \leq h(|v|)$ and $\int_0^\infty \frac{1}{h(v)} dv = +\infty$ and $h$ increasing.
- Let $T_n = \inf\{t > 0 : |v(t)| = 2^n\}$. 

So the times required to double have the property that $P(T_{n+1} - T_n) = +\infty$. Cannot explode in finite time.
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- \( 2^{n+1} = v(T_{n+1}) = v(T_n) + \int_{T_n}^{T_{n+1}} f(v(s)) ds \).

- This is enough to prove that solutions cannot explode in finite time.
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  \[ \leq 2^n + (T_{n+1} - T_n)h(2^{n+1}). \]
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\[
\sum_n \frac{2^n}{h(2^{n+1})} = +\infty \iff \int_1^\infty \frac{2^x}{h(2^x)} dx = +\infty \iff \int_0^\infty \frac{1}{h(x)} dx = +\infty.
\]
Idea of the proof

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- This is enough to prove that solutions cannot explode in finite time.

- So the times required to double have the property that $\sum (T_{n+1} - T_n) = +\infty$. Cannot explode in finite time.
Proof of global solutions

- Let $\tau_n = \inf\{t > 0 : |u(t)|_{L^\infty} > 3^n\}$. 
Proof of global solutions

- Let $\tau_n = \inf\{t > 0 : |u(t)|_{L^\infty} > 3^n\}$.
- Let $a_n = \min\left\{\frac{3^{n-1}}{h(3^n)}, \frac{1}{n}\right\}$. Notice that $\sum a_n = +\infty$. 
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- Goal of the proof: Show that eventually $\tau_{n+1} - \tau_n \geq a_n$ for all large $n$. 

For any fixed $n$ and $t > 0$, 

$u(\tau_n + t) = S(t)u(\tau_n) + \int_{\tau_n}^{\tau_n+t} S(t-s)f(u(s))ds + \int_{\tau_n}^{\tau_n+t} S(t-s)\sigma(u(s))dw(s).$ 

For $t \in [0, a_n + 1 \wedge (\tau_n + 1 - \tau_n)]$,

$|S(t)u(\tau_n)|_{L^\infty} \leq |u(\tau_n)|_{L^\infty} \leq 3^n.$

$\int_{\tau_n}^{\tau_n+t} S(t-s)f(u(s))ds_{L^\infty} \leq a_n + 1 \leq 3^n.$

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  \]
- For $t \in [0, a_{n+1} \wedge (\tau_{n+1} - \tau_n)]$
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$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \int_{\tau_n}^{\tau_n+t} S(\tau_n + t - s)\sigma(u(s))dw(s).$$

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$$|S(t)u(\tau_n)|_{L^\infty} \leq |u(\tau_n)|_{L^\infty} \leq 3^n.$$

$$\left| \int_{\tau_n}^{\tau_n+t} S(\tau_n + t - s)f(u(s))ds \right|_{L^\infty} \leq a_{n+1}h(3^{n+1}) \leq 3^n.$$
Proof of global solutions

- The only way that \((\tau_{n+1} - \tau_n)\) can be less than \(a_{n+1}\) is if

\[
\sup_{t \in [0, a_{n+1} \wedge (\tau_{n+1} - \tau_n)]} \left| \int_{\tau_n}^{\tau_n+t} S(\tau_n + t - s) \sigma(u(s)) dw(s) \right|_{L^\infty} > 3^n.
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\[
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\leq \mathbb{P} \left( \sup_{t \in [0, a_{n+1} \wedge (\tau_{n+1} - \tau_n)]} \left| \int_{\tau_n}^{\tau_n + t} S(\tau_n + t - s)\sigma(u(s))dw(s) \right|_{L^\infty} > 3^n \right)
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- Chebyshev

\[
\leq 3^{-np} \mathbb{E} \sup_{t \in [0, a_{n+1} \wedge (\tau_{n+1} - \tau_n)]} \left| \int_{\tau_n}^{\tau_n + t} S(\tau_n + t - s)\sigma(u(s))dw(s) \right|_{L^\infty}^p.
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\]

- By the moment bounds and \(\sigma(u(s)) \leq 3^{(n+1)(1-\gamma)}(h(3^{n+1})\gamma),\)

\[
\leq 3^{-np} \frac{p(1-\eta-\zeta)}{2} 3^p(1-\gamma)(n+1)(h(3^{n+1}))\gamma p \leq Ca_{n+1}^{ \frac{p(1-\eta-\zeta-2\gamma)}{2}}
\]
Proof of global solution

\[ \mathbb{P}(\tau_{n+1} - \tau_n \leq a_{n+1}) \leq C a_{n+1}^{\frac{p(1-\eta-\zeta-2\gamma)}{2}} \leq C n^{-\frac{p(1-\eta-\zeta-2\gamma)}{2}}. \]
Proof of global solution

\[ P(\tau_{n+1} - \tau_n \leq a_{n+1}) \leq Ca_{n+1}^{\frac{p(1-\eta-\zeta-2\gamma)}{2}} \leq Cn^{-\frac{p(1-\eta-\zeta-2\gamma)}{2}}. \]

Choose \( \zeta \) small enough and \( p \) large enough to that exponent \( \frac{p(1-\eta-\zeta-2\gamma)}{2} > 1 \).
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\[ P(\tau_{n+1} - \tau_n \leq a_{n+1}) \leq Ca_{n+1}^{p(1-\eta-\zeta-2\gamma)/2} \leq Cn^{-p(1-\eta-\zeta-2\gamma)/2}. \]

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By the Borel-Cantelli Lemma

\[ \sum_n P(\tau_{n+1} - \tau_n \leq a_{n+1}) \leq \sum_n Cn^{-q} \text{ for some } q > 1. \]
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\[ \mathbb{P}(\tau_{n+1} - \tau_n \leq a_{n+1}) \leq Ca_{n+1}^{\frac{p(1-\eta-\zeta-2\gamma)}{2}} \leq Cn^{-\frac{p(1-\eta-\zeta-2\gamma)}{2}}. \]

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- With probability one \( \tau_{n+1} - \tau_n > a_n \) for all large \( n \).

- Therefore \( P(\sup_n \tau_n = +\infty) = 1 \).