Global solutions to the stochastic reaction-diffusion equation with superlinear forcing and superlinear multiplicative noise

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Stochastic reaction-diffusion equation

Bounded spatial domain $x \in D \subset \mathbb{R}^d$.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathcal{A}u(t,x) + f(u(t,x)) + \sigma(u(t,x))\dot{w}(t,x) \quad t > 0, \quad x \in D, \\ u(t,x) = 0, \quad x \in \partial D \end{cases}$$

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- Nonlinear reaction term $f(u(t,x))$
- Multiplicative noise term $\sigma(u(t,x))\dot{w}$.

Introduction to stochastic PDE

- The solution to the non-stochastic heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$, $u(0,x) = u_0(x)$ can be written as a convolution with a heat kernel

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- If the domain is the whole space then $K(t,x,y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}}$.
- Duhamel's principle gives the solution to the semilinear heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + f(u)$

$$u(t,x) = \int_D K(t,x,y) u_0(y) dy + \int_0^t \int_D K(t-s,x,y) f(u(s,y)) dy ds.$$

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- Formally, solution should be given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds + \int_0^t S(t-s)\sigma(u(s))dw(s)$$

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  global solutions exist. (Brzezniak Peszat 1999, Iwata 1999, Cerrai 2003, Da Prato Röckner 2002, Marinelli Röckner 2010, Röckner Liu 2010)
- Mueller and collaborators (1991, with Sowers 1993, 1998, 2000) showed that in the space-time white noise case with $\mathcal{A} = \frac{\partial^2}{\partial x^2}$, $f \equiv 0$ and $\sigma(u) = |u|^\gamma$, solutions never explode if $\gamma < \frac{3}{2}$ and explode with positive probability if $\gamma > \frac{3}{2}$.

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- Dissipative forces push solutions away from $\pm\infty$, counteracting expansion due to noise.
- I outline some recent results about sufficient conditions in the accretive and dissipative settings that guarantee that mild solutions to the SRDE do not explode.

Technical Assumptions on $\mathcal{A}$ and $\dot{w}$

- Eigenvalues of $\mathcal{A}$, $\{e_k(x)\}$ complete orthonormal basis of $L^2(D)$.

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- Formal definition of the noise: Sequence $\lambda_j \geq 0$

$$\dot{w}(t,x) = \sum_{j=1}^{\infty} \lambda_j e_j(x) d\beta_j(t), \quad \beta_j(t) \text{ i.i.d. one-dimensional B.M.}$$

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- Condition (Cerrai 2003) - There exist $\theta > 0$, $\rho \in [2, +\infty]$

$$\sum_{k=1}^{\infty} \alpha_k^{-\theta} |e_k|_{L^\infty}^2 < +\infty, \;\; \sum_{j=1}^{\infty} \lambda_j^{\rho} |e_j|_{L^\infty}^2 < +\infty \text{ or } \sup_j \lambda_j < +\infty \; (\rho = \infty)$$

$$\eta := \frac{\theta(\rho - 2)}{\rho} < 1 \qquad (\eta := \theta \text{ if } \rho = +\infty)\,.$$

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- Space Hölder continuity $\approx (1 - \eta)$. Time Hölder continuity $\approx \frac{1-\eta}{2}$.

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- If $d > 1$, space-time white noise not allowed. On rectangular domains we need $\theta > \frac{d}{2}$, $\rho < \frac{2d}{d-2}$.

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- Generalization to higher spatial dimensions and other more general conditions in dissertation by Bezdek.

Polynomially dissipative f – SDE example

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- Similar conditions identified by Liu and Röckner (2010) for monotone SPDEs

$$2_{V^*}\left\langle A(t,v), v\right\rangle_V + \|B(t,v)\|_2^2 + \theta\|v\|_V^\alpha \leq C + K\|v\|_H^2.$$

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- When η not trace-class, Ito formula arguments are not available.
- As long as $\beta > 1$, γ can grow superlinearly.

Idea of the proof

- Set up a sequence of stopping times

$$\tau_0 = \inf\{t \geq 0 : |u(t)|_{L^\infty(D)} = 3^n c_0 \text{ for some } n \in \{1, 2, 3, ...\}\},$$

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- Eventually, $|u(\tau_k)|_{L^\infty}$ decreases or it takes more than $1/k$ time to triple. If it explodes it has more up steps than down steps.

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$$\mathbb{P}\left(|u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty} \text{ and } \tau_{k+1} - \tau_k < \varepsilon\right) \leq C\varepsilon^q.$$

- By Borel-Cantelli setting $\varepsilon = \frac{1}{k}$,

$$|u(\tau_{k+1})|_{L^\infty} = 3|u(\tau_k)|_{L^\infty} \text{ and } \tau_{k+1} - \tau_k < \frac{1}{k} \text{ a finite number of times.}$$

- Eventually, $|u(\tau_k)|_{L^\infty}$ decreases or it takes more than $1/k$ time to triple. If it explodes it has more up steps than down steps.
- This is enough to prove that $\sum_k (\tau_{k+1} - \tau_k) = +\infty$. Cannot explode in finite time.

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- The mild solution to the SPDE

$$\begin{aligned} u(t+\tau_k) =& S(t)u(\tau_k) + \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s) f(u(s))ds \\ &+ \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)\sigma(u(s))dW(s) \end{aligned}$$

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- If $Z_k(t) := \int_{\tau_k}^{\tau_k+t} S(\tau_k + t - s)\sigma(u(s))dW(s)$ satisfies $|Z_k(t)|_{L^\infty} \leq \frac{1}{3}|u(t+\tau_k)|_{L^\infty}$, for $t \in [0, \tau_{k+1} - \tau_k]$,

$$|u(t)|_{L^\infty} \leq \frac{3}{2}\left(|u(\tau_k)|_{L^\infty}^{-(\beta-1)} + Ct\right)^{-\frac{1}{\beta-1}}.$$

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- Moment bounds (Cerrai 2003) – For $p > 1$ large enough, and small numbers $\alpha, \gamma \in (0, (1-\eta)/2)$ such that $(\alpha - \frac{\zeta}{2})p > 1$ and

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- When $2\alpha + \eta + \frac{2(\gamma-1)}{\beta-1} < 1$, this inner integral is bounded as $t \downarrow 0$. (Beta function)

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Example

Theorem (S. 2022 – same theorem)

Assume $f(u)\text{sign}(u) \leq -\mu|u|^\beta$ *for* $|u| > c_0$, $|\sigma(u)| \leq C(1+|u|^\gamma)$, *with* $\gamma < 1 + \frac{(1-\eta)(\beta-1)}{2}$. *Then mild solutions can never explode.*

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- η can be chosen arbitrarily close to $\frac{1}{2}$ so if $\gamma < 1 + \frac{\beta-1}{4}$ then solutions cannot explode in finite time.
- If $\beta \leq 3$, the Mueller results give a stronger result, so γ can be any value $\gamma < \frac{3}{2}$.
    - The semigroup causes $t^{-\frac{1}{2}}$ decay but the nonlinear term causes $t^{-\frac{1}{\beta-1}}$ decay.

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- Explodes in finite time if and only if $\int_c^\infty \frac{1}{f(x)} dx < +\infty$.

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Osgood condition for SRDE $\frac{\partial u}{\partial t} = \mathcal{A}u + f(u) + \sigma(u)\dot{W}$

- For deterministic PDEs, Osgood condition does not fully characterize explosion (Fujita 1966). $\frac{\partial u}{\partial t} = \Delta u + |u|^p$ if $p > 1 + \frac{2}{d}$.

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- Dalang, Khoshnevisan, and Zhang (2019) showed that σ can be superlinear too. Studied the space-time white noise on a bounded one-dimensional spatial domain
  - Assume $|f(u)| \leq C(1 + |u| \log |u|)$, $\sigma \in o(|u|(\log |u|)^{\frac{1}{4}})$.
  - Then solutions never explode.

Result for accretive f

Theorem (S. 2022)

Assume f and σ are locally Lipschitz continuous functions. Assume that there exists a positive, increasing function $h : [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^\infty \frac{1}{h(u)} du = \infty \text{ and}$$

$$|f(u)| \leq h(|u|)$$

and there exists $\gamma \in \left(0, \frac{1-\eta}{2}\right)$ such that

$$|\sigma(u)| \leq |u|^{1-\gamma}(h(|u|))^\gamma \text{ for all } |u| > 1.$$

Assume the initial data is bounded. Then there exists a unique global solution.

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and there exists $\gamma \in \left(0, \frac{1-\eta}{2}\right)$ such that

$$|\sigma(u)| \leq |u|^{1-\gamma}(h(|u|))^\gamma \text{ for all } |u| > 1.$$

Assume the initial data is bounded. Then there exists a unique global solution.

If $h(u) = u \log(u) \log\log(u)$ then $\sigma(u) \leq u \left(\log u \log\log u\right)^\gamma$.

Local mild solution and explosion

- Define the cutoff versions of f and σ

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in [-3^n, 3^n] \\ f(3^n) & \text{if } x > 3^n \\ f(-3^n) & \text{if } x < -3^n \end{cases}$$

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- Solution EXPLODES in finite time if $\sup_n \tau_n < +\infty$ and solution is global in time if $\sup_n \tau_n = +\infty$.

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- For any $\zeta \in (0, 1-\eta)$, $p > \max\left\{\frac{2}{1-\eta-\zeta}, \frac{d}{\zeta}\right\}$, there exists $C = C(\zeta, p)$ such that for any adapted random field that is almost surely bounded

$$\mathbb{P}\left(\sup_{s \in [0,t]} \sup_{x \in D} |\Phi(t,x)| \leq M\right) = 1.$$

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- One interpretation is that time Hölder continuity is slightly worse than $\frac{1-\eta}{2}$ – same number that describes how superlinear σ can be.

Idea of the proof

- Consider the ODE $\frac{dv}{dt} = f(v(t))$ where $|f(v)| \leq h(|v|)$ and $\int_0^\infty \frac{1}{h(v)} dv = +\infty$ and h increasing.

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- So the times required to double have the property that $\sum (T_{n+1} - T_n) = +\infty$. Cannot explode in finite time.

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- The only way that $(\tau_{n+1} - \tau_n)$ can be less than a_{n+1} is if

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- By the moment bounds and $\sigma(u(s)) \leq 3^{(n+1)(1-\gamma)}(h(3^{n+1})^\gamma$,

$$\leq 3^{-np} a_{n+1}^{\frac{p(1-\eta-\zeta)}{2}} 3^{p(1-\gamma)(n+1)}(h(3^{n+1}))^{\gamma p} \leq C a_{n+1}^{\frac{p(1-\eta-\zeta-2\gamma)}{2}}$$

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- Therefore $\mathbb{P}\left(\sup_n \tau_n = +\infty\right) = 1$.

Thank you