Diffusion processes in random environments on disconnected selfsimilar fractal sets in $\mathbb{R}$

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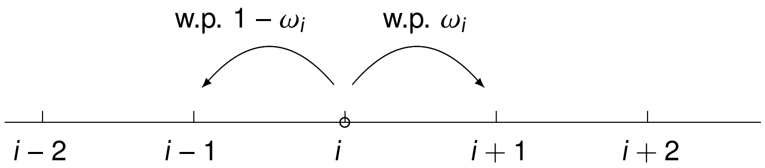
based on a joint work with Y. TAMURA (Keio Univ.)

Sinai's walk (random walks in random environments)

$\{\omega_i, i \in \mathbb{Z}, Q\}$: $\omega_i \in (0,1)$, i.i.d. r.v.s



$\sigma_i := \omega_i/(1-\omega_i)$: independent from random walks

Theorem (Sinai 1982)

If we assume that $E[\log \sigma_i] = 0$ and $E\left[(\log \sigma_i)^2\right] < \infty$,
then the distribution of $\left\{(\log n)^{-2} X_n\right\}$ converges
to some functional of random environments.

(Ultra-slow diffusive property)

Brox's diffusion on $\mathbb{R}$

Random environment

$\mathbb{W}$: the space of continuous functions $W : \mathbb{R} \to \mathbb{R}$.
Q: the Wiener measure on $\mathbb{W}$.
$\{W(x), x \geq 0, Q\}$ and $\{W(-x), x \geq 0\}$ are
independent one-dimensional Brownian motions starting at 0.

- $W = \{W(x), x \in \mathbb{R}, Q\}$: a Brownian environment.

Diffusion process in Brownian environment

Given a Brownian environment, Brox considered
a diffusion process starting at 0 with generator

$$\frac{1}{2} e^{W(x)} \frac{d}{dx} \left(e^{-W(x)} \frac{d}{dx} \right).$$

- $X(t, W)$: a diffusion process in a Brownian environment.

Construction of $X(t, W)$

Ω : the space of continuous paths $\omega : [0, \infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$.
$\omega(t)$: the value of a function $\omega \in \Omega$ at time t.

For a fixed W we consider a probability measure P_W on Ω s.t. $\{\omega(t), t \geq 0, P_W\}$ is a diffusion process with generator

$$\frac{1}{2} e^{W(x)} \frac{d}{dx} \left(e^{-W(x)} \frac{d}{dx} \right).$$

A version of $X(t, W)$

$B(t)$: a one-dimensional Brownian motion starting at 0.
$L(t, x)$: a local time of $B(t)$ at x.
$S(x) = \int_0^x e^{W(y)}\, dy$: scale function.
$A(t) = \int_0^t e^{-2W(S^{-1}(B(s)))}\, ds = \int_{\mathbb{R}} e^{-2W(S^{-1}(x))} L(t, x)\, dx$: time change.
$X(t, W) = S^{-1}(B(A^{-1}(t)))$.

Brox's result

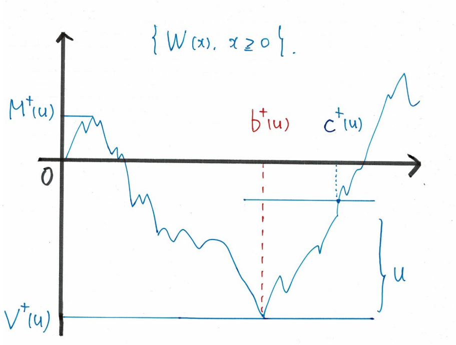
• Assume that (W, Q) and (Ω, P_W) are independent.
$\Longrightarrow$ The full distribution governing $X(t)$ is $\mathcal{P} = P_W \otimes Q$.

Theorem (Brox 1986)
The distribution of $\left\{(\log t)^{-2} X(t)\right\}$ converges to some functional of W.

Remark (Kawazu-Tamura-Tanaka 1989)
Sinai's result can be derived from the diffusion case.

The notion of "a valley" plays an important role
in Brox's diffuision and Sinai's walk.

Description of a Valley 1/2



$$W^\sharp(x) := W(x) - \min_{y\in[0,x]} W(y), x \geq 0.$$

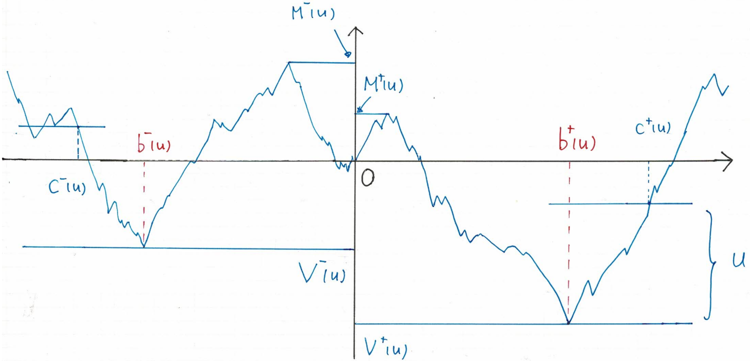
$$c^+(u) := \inf\left\{x > 0 : W^\sharp(x) = u\right\}.$$

$$V^+(u) := \min_{y\in[0,d^+(u)]} W(y).$$

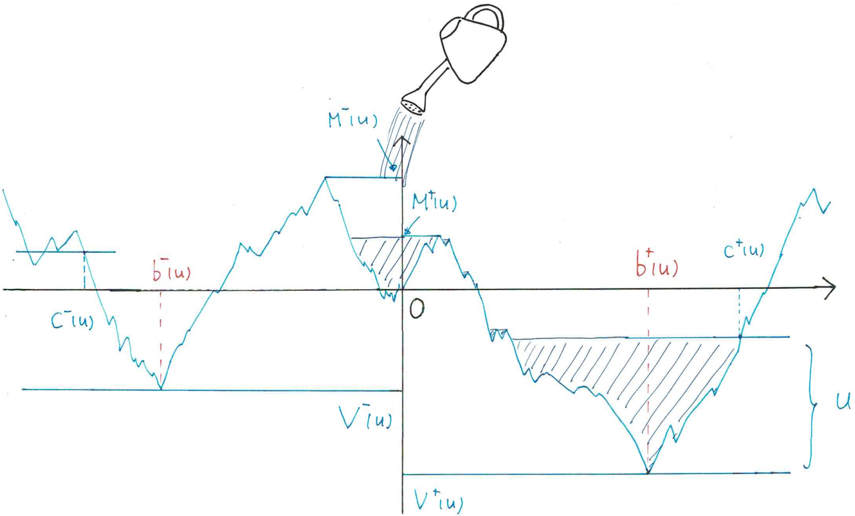
$$b^+(u) \text{ s.t. } W(b^+(u)) = V^+(u).$$

$$M^+(u) := \max_{y\in[0,b^+(u)]} W(y).$$

Description of a Valley 2/2



$$b(u,W) := \begin{cases} b^{+}(u) & \\ \qquad \text{if } \max\{M^{+}(u),(V^{+}(u)+u)\} < \max\{M^{-}(u),(V^{-}(u)+u)\}, \\ b^{-}(u) & \\ \qquad \text{if } \max\{M^{+}(u),(V^{+}(u)+u)\} > \max\{M^{-}(u),(V^{-}(u)+u)\}. \end{cases}$$



$$b(u, W) = b_+(u)$$

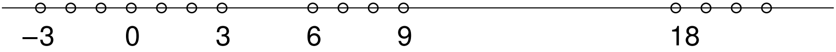
Random walks on a pre-Cantor set F_0

- Let us consider Brox-type diffusions on disconnected fractal set in $\mathbb{R}$.

$\Longrightarrow$ "A Brownian motion on a disconnected fractal set" constructed by scaled random walks.

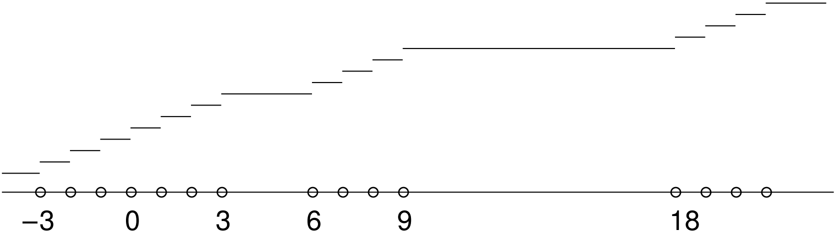
Random walk on pre-Cantor set F_0 (Triadic Cantor set)

- Consider one-dimensional simple random walks starting from the origin.
- Observe the random walks on $F_0 \Longrightarrow$ denote by R_n.



- Find suitable scale and time changes to obtain a weak convergence to a stochastic process.

Continuous time random walks on F_0



$m_0(x)$: pre-Cantor function s.t. $m_0(dx) := \sum_{i \in F_0} \delta_i(dx)$.

$B(t) := \omega(t)$: Brownian motion starting at 0, $\{B(t), t \geq 0, P\}$.
$L(t, x)$: a local time of $B(t)$ at x.

$$A_0(t) := \int_{\mathbb{R}} L(t, x) m_0(dx).$$

$R(t) := B(A_0^{-1}(t))$: continuous time R.W. on F_0.

$m_c(x) := \lim_{n \to \infty} \frac{1}{2^n} m_0(3^n x)$: an infinitely extended selfsimilar measure.

(infinitely extended triadic Cantor function)

Disconnected selfsimilar fractal sets in $\mathbb{R}$ and their selfsimilar measures

Mapping $r > 1, \varphi = (\varphi_1, \ldots, \varphi_N)$: a family of r-similitudes on $[0,1]$.

Assumptions $\varphi_1(x) = x/r,\ \varphi_N(x) = x/r + (1 - 1/r)$,
$\varphi([0,1]) \bigcap \varphi_j([0,1]) = \emptyset, i \neq j$. (Open set condition)

$\Longrightarrow$ There exists a unique compact set $\widetilde{C} \subset [0,1]$ s.t.

$$\widetilde{C} = \bigcup_{i=1}^{N} \varphi_i\left(\widetilde{C}\right).$$

For $\widetilde{C}$, we set a selfsimilar measure s.t. $\widetilde{m}(\widetilde{C}) = 1$ and

$$\widetilde{m}(A) = \frac{1}{N} \sum_{i=1}^{N} \widetilde{m}(\varphi_i^{-1}(A)) \text{ for any Borel set } A \subset [0,1].$$

Example The triadic Cantor set: $r = 3,\ N = 2$,

$\varphi_1(x) = x/3,\ \varphi_2(x) = x/3 + 2/2,\ \widetilde{m}$: the triadic Cantor function.

Unbounded cases

Bounded fractal set

$\widetilde{F}_0 = \{0,1\},\ \widetilde{F}_{n+1} = \bigcup_{i=1}^{N} \varphi_i(\widetilde{F}_n),\ \widetilde{F}_\infty = \bigcup_{n=0}^{\infty} F_n, \qquad \widetilde{C} := Cl(\widetilde{F}_\infty).$

Unbounded fractal set

$F_0 = \bigcup_{n=0}^{\infty} \varphi_1^{-n}(\widetilde{F}_n),\ F_n = \varphi_1^n(F_0),\ F_\infty = \bigcup_{n=0}^{\infty} F_n, \qquad C^+ := Cl(F_\infty).$

$C := C^+ \cup (-C^+)$: unbounded dis-connected selfsimilar fractal set.

Unbounded selfsimilar measure

$m^+(A) = N^n \widetilde{m}(\varphi_1^n(A)),\ A \subset [0, r^n]$

$m^-(A') = -m^+(-A'),\ A' \subset [-r^n, 0]$

$m_c(x) = \begin{cases} m^+(x), & x \geq 0, \\ m^-(x), & x < 0. \end{cases}$: infinitely extended selfsimilar measure.

$$m_c(x) = \frac{1}{N^n} m_c(r^n x),\ x \in \mathbb{R}.$$

Scaled limit process of random walks

Theorem (T.-Tamura)
Scaled random walk on F_0, $\left\{\frac{1}{r^n}R((rN)^n t)\right\}$ converges weakly to a generalized diffusion process whose generator is given by $\frac{1}{2}\frac{d}{dm_c(x)}\frac{d}{dx}$.

- $B_c(t)$ denotes the generalized process above.

A version of $B_c(t)$

$B(t)$: a one-dimensional Brownian motion starting at 0.

$A_c(t) := \int_{\mathbb{R}} L(t,x) dm_c(x)$.

$B_c(t) := B(A_c^{-1}(t))$.

Semi-selfsimilarity: $\{B_c(t), t \geq 0\} \overset{d}{=} \left\{\frac{1}{r^n}B_c((rN)^n t), t \geq 0\right\}$,

where $\overset{d}{=}$ means the equality in distribution with respect to P.

Diffusion processes on disconnected fractal sets

Remark Fujita studied the growth order of eigenvalues of the generator $\frac{d}{dm_c}\frac{d}{dx}$ (1987, 1990).

Remark Golmankhaneh et al. (2018)
Simulation studies on fractional equations of diffusions on Cantor sets
Simple random walks' case: scaling property $\left\{\frac{1}{r^n}R([r^{2n}t])\right\}$

"Diffusion process" on the triadic Cantor set moves quickly.
(Super-diffusive property)

α-stable Lévy process $\{Y(t)\}$
Selfsimilarity of α-stable Lévy process:

$$\{Y(t)\}\stackrel{\mathrm{d}}{=}\left\{\frac{1}{\lambda}X(\lambda^{\alpha}t)\right\} \text{ for any } \lambda > 0.$$

$\alpha = 2$: diffusive (Brownian motion); $\alpha \in (0, 2)$: super-diffusive.

Brox's diffusion on the Cantor set

Random environment

(W, Q): a Brownian environment..
$m_c(x) = \frac{1}{N} m_c(rx)$: an infinitely extended selfsimilar measure
$W_c(x) := W(m_c(x))$ is semi-selfsimilar s.t.

$$\{W_c(x), x \in \mathbb{R}\} \overset{\mathcal{D}}{=} \left\{ \frac{1}{N} W_c(r^2 x), x \in \mathbb{R} \right\},$$

where $\overset{\mathcal{D}}{=}$ means the equality in distribution with respect to Q.

Fixing W_c, we consider a diffusion process starting from zero whose generator is given by

$$L_{W_c} = \frac{1}{2} \exp\{W_c(x)\} \frac{d}{dm_c} \left(\exp\{-W_c(x)\} \frac{d}{dx} \right).$$

- $X(t, W_c)$: Brox diffusion process on C starting from zero.

Main theorem

A version of $X(t, W_c)$

$B_c(t)$: a Brownian motion on C.

$$S(x) = \int_0^x e^{W_c(y)}\, dy,$$

$$A(t) = \int_{\mathbb{R}} e^{-2W_c(S^{-1}(x))} L(t,x) dm_c(x) = \int_0^t e^{-2W_c(S^{-1}(B_c(s)))}\, ds,$$

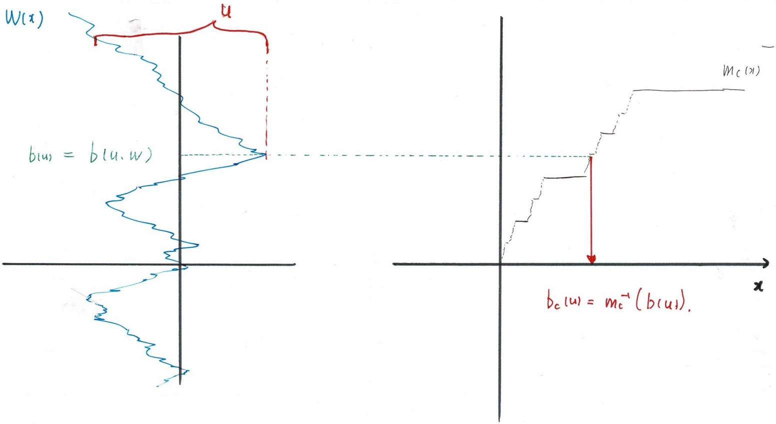
$$X(t, W_c) = S^{-1}(B_c(A^{-1}(t))).$$

The full distribution governing $\{X(t), t \geq 0\}$ is $\mathcal{P}_c = P_{W_c} \otimes Q$.

Theorem

Any finite-dimensional distribution of the process $\left\{ \frac{1}{r^{2n}} X(e^{N^n u}), u > 0, \mathcal{P}_c \right\}$ converges as $n \to \infty$ to the corresponding finite-dimensional distribution of the process $\{b_c(u), u > 0, Q\}$, where $b_c(u) = m_c^{-1}(b(u, W))$.

Description of $b_c(u)$



$$b_c(u) = m_c^{-1}(b(u, W)).$$

Remark

For Q-almost all environments, $b_c(u, W_c)$ is uniquely determined with a fixed $u > 0$.

Scaling property

For $n \in \mathbb{Z}$, we set

$$W_c^{(n)} = \frac{1}{N^n} W_c(r^{2n} x).$$

Lemma For each n and W_c,

$$\left\{X\left(t, N^n W_c^{(n)}\right), t \geq 0\right\} \stackrel{d}{=} \left\{\frac{1}{r^{2n}} X((rN)^{2n} t, W_c), t \geq 0\right\}.$$

- To show the convergence in finite distributions

$$\left\{\frac{1}{r^{2n}} X(e^{N^n u}), u > 0, \mathcal{P}_c\right\} \stackrel{f.d.}{\Longrightarrow} \{b_c(u), u > 0, Q\},$$

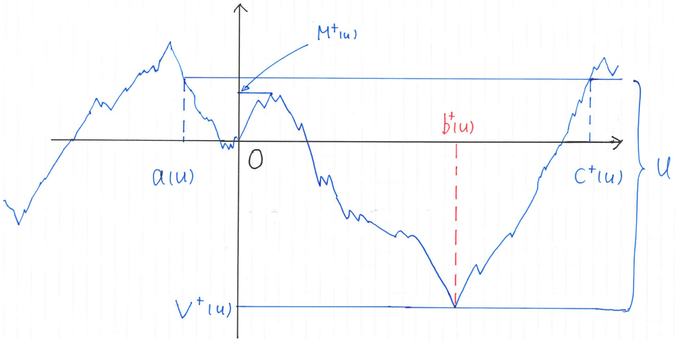
we study limiting behavior of $X\left(t, N^n W_c\right)$.

- For simplicity, we consider the case $b(u, W) = b_+(u)$.

Case (I) $V^{+}(u) + u > M^{+}(u)$.

$a(u) = a(u, W) := \sup\{x < 0 : W(x) = W(c^{+}(u))\}$.

$(a(u), b^{+}(u), c^{+}(u))$: a valley of W containing 0 with the depth u.



$a_c := m_c^{-1}(a), c_c := m_c^{-1}(c)$.

Case (I)

$$X_n(t,u,W_c) := \frac{1}{r^{2n}} X(e^{N^n u} t, W_c) - b_c(u,W_c).$$

$$Y_n(t,u,W_c) := X((rN)^{-2n} e^{N^n u} t, N^n W_c) - b_c(u,W_c).$$

The scaling property implies that

$$\{X_n(t,u,W_c), t \geq 0\} \overset{\mathcal{L}}{=} \{Y_n(t,u,W_c), t \geq 0\}.$$

where $\overset{\mathcal{L}}{=}$ means the equality in distribution with respect to $\mathcal{P}_c$.

- For a fixed $u > 0$ and W_c, the generator of $Y_n(t,u,W_c)$

$$\widehat{L}_n = \frac{e^{n^n u}}{2(rN)^{2n}} \exp\{N^n(W_c(x+b_c(u)) - W_c(b_c(u)))\} \frac{d}{dm_c} \left\{ \exp\{-N^n(W_c(x+b_c(u)) - W_c(b_c(u)))\} \frac{d}{dx} \right\}.$$

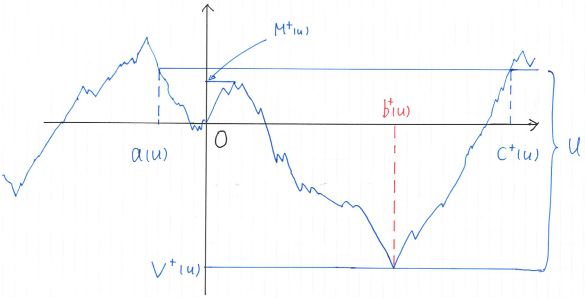
A property of scale function

Constant: $\epsilon_1 < \min\{b_c, (c_c - b_c)\}/2, \quad l_c := a_c - b_c, \quad r_c := c_c - b_c.$

$$\widehat{S}_n(x) = \frac{2(rN)^{2n}}{e^{N^n u}} \int_0^x e^{N^n\{W_c(y+b_c)-W_c(b_c)\}}\,dy \cdot \int_{-\epsilon_1}^{\epsilon_1} e^{-N^n\{W_c(y+b_c)-W_c(b_c)\}}\,dm_c.$$

<u>Proposition</u>

$$\widehat{S}_n(x) \to \begin{cases} -\infty, & x < l_c, \\ 0, & x \in (l_c, r_c), \\ \infty, & x > r_c. \end{cases}$$

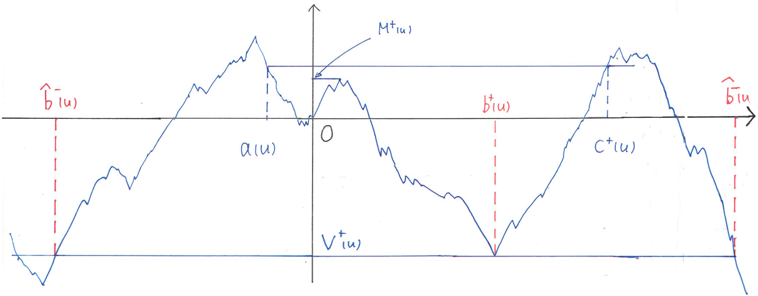


A property of speed measure

$$\widehat{m}_n(x) = \int_0^x e^{-N^n\{W_c(y+b_c)-W_c(b_c)\}}\, dm_c \cdot \left\{ \int_{-\epsilon_1}^{\epsilon_1} e^{-N^n\{W_c(y+b_c)-W_c(b_c)\}}\, dm_c \right\}^{-1}.$$

<u>Proposition</u>

$$\widehat{m}_n(x) \to \begin{cases} -\infty, & x < \hat{b}^-, \\ 0, & x \in (\hat{b}^-, 0), \\ 1, & x \in (0, \hat{b}^+), \\ \infty, & x > \hat{b}^+. \end{cases}$$



Convergence theorem by Ogura

Bi-generalized diffusion processes introduced by Ogura (1989).

State space: $Q := (l_c, r_c)$.
We set a measure $\widehat{M}_n = \widehat{m}_n \circ (\widehat{S}_n)^{-1}$.
$Y_n(t)$: the generalized diffusion process associated with $(\widehat{S}_n, \widehat{m}_n)$.
Then, $\widehat{M}_n(dx)$ converges vaguely to the delta measure on Q.
$Z(t)$: the generalized diffusion process associated with $(x, \delta(dx))$.
starting from a point in Q.

- $Z(t)$ jumps to zero immediately at $t = 0$ and remains there for any $t > 0$.
- Ogura's convergence theorem implies that for each $u > 0$ and $t > 0$

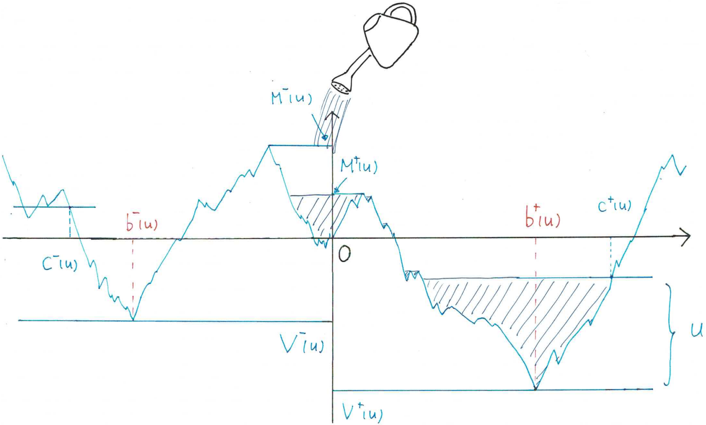
$$\lim_{n\to\infty} P\{|Y_n(t, u, W_c)| < \epsilon\} = 1.$$

- Scaling property implies that for the ϵ-neighborhood of the point $b_c(u)$

$$\mathcal{P}\left\{\frac{1}{r^{2n}} X(e^{N^n u}) \in U\right\} \to 1 \text{ as } n \to \infty.$$

Case (II)

Case (II) $V^{+}(u) + u < M^{+}(u)$.



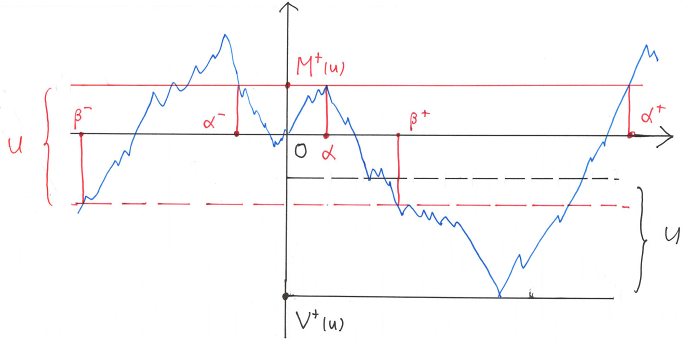
$b(u, w) = b_{+}(u)$

Case (II)

Set $\alpha = \alpha(u)$ s.t. $W(\alpha) = M^+(u)$.

$\alpha^+(u) = \inf\{x > \alpha : W(x) = M^+(u)\}$, $\alpha^-(u) = \sup\{x < 0 : W(x) = M^+(u)\}$,
$\beta^+(u) = \inf\{x > 0 : W(x) = M^+(u) - u\}$,
$\beta^-(u) = \sup\{x < 0 : W(x) = M^+(u) - u\}$.

Using m_c, we set $\alpha_c, \alpha_c^\pm$ and $\beta_c^\pm$.



Generator

Consider the process,

$$Z_n(t,u,W_c) = X((rN)^{-2n} e^{N^n u} t, N^n W_c)$$

whose generator is given by

$$\widetilde{L}_n = \frac{e^{N^n u}}{2(rN)^{2n}} \exp\{N^n W_c\} \frac{d}{dm_c} \left\{ \exp\{-N^n W_c\} \frac{d}{dx} \right\}.$$

- The scale function of the diffusion process $\widetilde{S}_n(x)$:

$$\widetilde{S}_n(x) = \int_0^x e^{N^n W_c(y)}\, dy \cdot \left\{ \int_{\alpha_c - \epsilon_2}^{\alpha_c + \epsilon_2} e^{-N^n W_c(y)}\, dy \right\}^{-1},$$

where ϵ_2 is a constant satisfying $\epsilon_2 < \min\{\alpha_c, (\beta_c^+ - \alpha_c)\}/2$.

- The speed measure $\widetilde{m}_n(dx)$:

$$\widetilde{m}_n(x) = e^{-N^n(W_c+u)}\, dm_c \cdot \frac{2(rN)^{2n}}{e^{N^n u}} \int_{\alpha_c - \epsilon_2}^{\alpha_c + \epsilon_2} e^{-N^n W_c}\, dy.$$

Convergence theorem for Case (II)

For a fixed $u > 0$, we obtain

$$\widetilde{S}_n(x) \to \begin{cases} -\infty, & x < \alpha_c^-, \\ 0, & \alpha_c^- < x < \alpha_c, \\ 1, & \alpha_c < x < \alpha_c^+, \\ \infty, & \alpha_c^+ < x, \end{cases} \quad \text{and } \widetilde{m}_n(x) \to \begin{cases} -\infty, & x < \beta_c^-, \\ 0, & \beta_c^- < x < \beta_c^+, \\ \infty, & \beta_c^+ < x, \end{cases}$$

as $n \to \infty$ for Q-almost all W_c.

$$\Longrightarrow \lim_{n\to\infty} P^0 \left\{ X((rN)^{-2n} e^{N^n u}, N^n W_c) < y \right\} = 0$$

for any $y \in (\alpha_c^-, \beta_c^+)$ and for almost all W_c.

- $X((rN)^{-2n} e^{N^n u}, N^n W_c)$ starting from zero immediately hits β_c^+.
- We again consider the process starting from β_c^+.

Appendix

Scaling property 1/3

For $n \in \mathbb{Z}$, we set

$$W_c^{(n)} = \frac{1}{N^n} W_c(r^{2n}x).$$

Lemma For each n and W_c,

$$\left\{X\left(t, N^n W_c^{(n)}\right), t \geq 0\right\} \overset{d}{=} \left\{\frac{1}{r^{2n}} X((rN)^{2n}t, W_c), t \geq 0\right\}.$$

Outline of proof

- The scale function of $X\left(t, N^n W_c^{(n)}\right)$

$$S_n(x) := \int_0^x \exp\{N^n W_c^{(n)}(y)\, dy = \frac{1}{r^{2n}} \int_0^{r^{2n}x} \exp\{W_c(u)\}\, du = \frac{1}{r^{2n}} S(r^{2n}x).$$

- Time change function, $A_n(t) = A_n(t, B_c)$

$$= \int_0^t \exp\{-2N^n W_c^{(n)}(S_n^{-1}(B_c(s)))\}\, ds = \int_0^t \exp\{-2W_c^{(n)}(S_n^{-1}(r^{2n}B_c(s)))\}\, ds$$

Scaling property 2/3

- Semi-selfsimilarity of $\{B_c(t)\}$

$$\{B_c(t), t \geq 0\} \stackrel{d}{=} \left\{\frac{1}{r^{2n}} B_c((rN)^{2n} t), t \geq 0\right\} =: \left\{B_c^{(n)}(t), t \geq 0\right\}.$$

$$\left\{S_n^{-1}(B_c(A_n^{-1}(t, B_c)))\right\} \stackrel{d}{=} \left\{S_n^{-1}(B_c^{(n)}(A_n^{-1}(t, B_c^{(n)})))\right\}.$$

- Time change function

$$\begin{aligned} A_n(t, B_c^{(n)}) &= \int_0^t \exp\{-2W_c^{(n)}(S_n^{-1}(r^{2n} B_c^{(n)}(s)))\}\, ds \\ &= \int_0^t \exp\{-2W_c^{(n)}(S_n^{-1}(B_c((rN)^{2n} s)))\}\, ds \\ &= \frac{1}{(rN)^{2n}} A((rN)^{2n} t, B_c). \end{aligned}$$

$$\Longrightarrow \qquad A_n^{-1}(t, B_c^{(n)}) = \frac{1}{(rN)^{2n}} A^{-1}((rN)^{2n} t, B_c).$$

Scaling property 3/3

$$\left\{S_n^{-1}(B_c(A_n^{-1}(t,B_c)))\right\} \stackrel{d}{=} \left\{S_n^{-1}(B_c^{(n)}(A_n^{-1}(t,B_c^{(n)})))\right\}.$$

$$\begin{aligned} & S_n^{-1}(B_c^{(n)}(A_n^{-1}(t,B_c^{(n)}))) \\ = \quad & \frac{1}{r^{2n}} S^{-1}\left(B_c((rN)^{2n} A_n^{-1}(t,B_c^{(n)}))\right) \\ = \quad & \frac{1}{r^{2n}} S^{-1}(B_c(A^{-1}((rN)^{2n}t,B_c) \\ = \quad & \frac{1}{r^{2n}} X((rN)^{2n}t,W_c). \end{aligned}$$