Variation of anticyclotomic Iwasawa invariants in Hida families

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Glenn Stevens' 60th Birthday

Outline

Big Heegner points in the definite setting

Higher weight theta elements

Two-variable anticyclotomic p-adic L-functions

The definite setting

- $K/\mathbb{Q}$ imaginary quadratic field.
- $N \geq 1$ integer, $(N, D_K) = 1$.
- Factor $N = N^+N^-$ with:

$$\begin{aligned} \ell | N^+ \quad &\Longrightarrow \quad \ell \text{ splits in } K; \\ \ell | N^- \quad &\Longrightarrow \quad \ell \text{ is inert in } K. \end{aligned}$$

Assume N^- is the square-free product of an *odd* number of primes.

- $B/\mathbb{Q}$ quaternion algebra ramified at ∞N^-.
- Fix $\mathbb{Q}_\ell$-algebra isomorphisms

$$\imath_\ell : B_\ell := B \otimes \mathbb{Q}_\ell \xrightarrow{\sim} M_2(\mathbb{Q}_\ell)$$

  for all $\ell \nmid \infty N^-$.
- Fix a prime $p \geq 5$, $p \nmid ND_K$.

"Hida varieties"

- $R_m \subset B$ Eichler order of level N^+p^m.
- $U_m \subset \hat{R}_m^\times$ compact open subgroup ($\hat{R}_m := R_m \otimes \hat{\mathbb{Z}}$):

$$U_m := \{(x_\ell)_\ell \in \widehat{R}_m^\times \;:\; \imath_p(x_p) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \pmod{p^m}\}.$$

- Definite Shimura curves:

$$X_m(\mathbb{C}) = B^\times \backslash (\hat{B}^\times \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, B_\infty))/\hat{R}_m^\times;$$
$$\widetilde{X}_m(\mathbb{C}) = B^\times \backslash (\hat{B}^\times \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, B_\infty))/U_m.$$

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{X}_m \arrow[r, "\widetilde{\alpha}_m"] \arrow[d] & \widetilde{X}_{m-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & X_m \arrow[r] & X_{m-1} \arrow[r] & \cdots \end{tikzcd}$$

Heegner points in the definite setting

For $c \geq 1$, $(c, N) = 1$, let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$.

Definition

- $P = [(b, \psi)] \in \widetilde{X}_m(K)$ is a *Heegner point of conductor c* if

$$\psi(\mathcal{O}_c) = (b^{-1}\hat{R}_m^\times b \cap B) \cap \psi(K)$$

  and

$$\psi_p((\mathcal{O}_c \otimes \mathbb{Z}_p)^\times \cap (1 + p^m\mathcal{O}_K \otimes \mathbb{Z}_p)^\times) = b_p^{-1} U_{m,p} b_p.$$

- Galois action: For $\sigma \in G_K$ and $P = [(b, \psi)] \in \widetilde{X}_m(K)$, define

$$P^\sigma := [(b\hat{\psi}(a_\sigma), \psi)],$$

  where $a_\sigma \in \widehat{K}^\times$ is such that $\mathrm{rec}_K(a_\sigma) = \sigma|_{K^{\mathrm{ab}}}$.

A careful construction

Theorem (Longo–Vigni, *d'après* Howard)

There exists a system of Heegner points $P_{c,m} \in \widetilde{X}_m(K)$ such that:

1. $P_{c,m} \in H^0(H_{cp^m}(\boldsymbol{\mu}_{p^m}), \widetilde{X}_m(K))$.
2. *(Galois equivariance) For all $\sigma \in G_{H_{cp^m}}$,*
$$P_{c,m}^{\sigma} = \langle \vartheta(\sigma) \rangle P_{c,m},$$
where $\vartheta : G_{H_{cp^m}} \longrightarrow \mathbb{Z}_p^{\times}/\{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$.
3. *(Vertical compatibility) If $m \geq 2$,*
$$\widetilde{\alpha}_m(\mathrm{tr}_{H_{cp^m}(\boldsymbol{\mu}_{p^m})/H_{cp^{m-1}}(\boldsymbol{\mu}_{p^m})}(P_{c,m})) = U_p \cdot P_{c,m-1}.$$
4. *(Horizontal compatibility) If $p|c$,*
$$\mathrm{tr}_{H_{cp^m}(\boldsymbol{\mu}_{p^m})/H_{cp^{m-1}}(\boldsymbol{\mu}_{p^m})}(P_{c,m}) = U_p \cdot P_{c/p,m}.$$

Hida–Hecke algebras

Let $f_o \in S_{k_o}(\Gamma_0(N))$ p-ordinary newform defined over $F/\mathbb{Q}_p$.

- $\mathfrak{h}_m$: full Hecke algebra over $\mathcal{O}_F$ acting on $S_2(\Gamma_0(N) \cap \Gamma_1(p^m), \overline{\mathbb{Q}}_p)$.
- $\mathfrak{h}_\infty^{\mathrm{ord}} := \varprojlim_m e^{\mathrm{ord}} \mathfrak{h}_m$.
- $\mathbb{T}_m$: quotient of $\mathfrak{h}_m$ acting faithfully on the N^--new part.
- $\mathbb{T}_\infty^{\mathrm{ord}} := \varprojlim_m e^{\mathrm{ord}} \mathbb{T}_m$.

f_o defines an $\mathcal{O}_F$-algebra homomorphism

$$\lambda_{f_o} : \mathfrak{h}_\infty^{\mathrm{ord}} \longrightarrow \mathcal{O}_F$$

factoring through $\mathbb{T}_\infty^{\mathrm{ord}}$.

- $\mathbb{I}$: the unique irreducible component of $(\mathfrak{h}_\infty^{\mathrm{ord}})_{\mathfrak{m}_o}$ containing $\ker(\lambda_{f_o})$.
- $\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ (branch of) the Hida family passing of f_o.

Big Heegner points in the definite setting

- Let $\mathbb{D}_m := e^{\mathrm{ord}}\mathrm{Div}(\widetilde{X}_m) \otimes_{\mathfrak{h}_\infty^{\mathrm{ord}}} \mathbb{I}$.
- By Galois equiv., the image $\mathbb{P}_{c,m}$ of $P_{c,m}$ in $\mathbb{D}_m^\dagger$ is fixed by $G_{H_{cp^m}}$.
- By vertical compatibility,

$$\widetilde{\alpha}_m(\mathrm{Cor}_{H_{cp^m}/H_c}(\mathbb{P}_{c,m})) = U_p \cdot \mathrm{Cor}_{H_{cp^{m-1}}/H_c}(\mathbb{P}_{c,m-1})$$

in $H^0(H_c, \mathbb{D}_m^\dagger)$.

Definition

The *big Heegner point of conductor c* is

$$\mathcal{P}_c := \varprojlim_m U_p^{-m} \cdot \mathrm{Cor}_{H_{cp^m}/H_c}(\mathbb{P}_{c,m})$$

in $H^0(H_c, \mathbb{D}^\dagger) = \varprojlim_m H^0(H_c, \mathbb{D}_m^\dagger)$.

Big theta elements

- K_∞/K anticyclotomic $\mathbb{Z}_p$-extension of K.
- $G_\infty = \mathrm{Gal}(K_\infty/K)$.
- Let $\mathbb{J} := \varprojlim_m \mathbb{J}_m$, where $\mathbb{J}_m := e^{\mathrm{ord}}\mathrm{Pic}(\widetilde{X}_m) \otimes_{\mathfrak{h}_\infty^{\mathrm{ord}}} \mathbb{I}$.

Assumption

$\dim_{\kappa_{\mathbb{I}}}(\mathbb{J}/\mathfrak{m}_{\mathbb{I}}\mathbb{J}) = 1$.

- Then $\mathbb{J}$ is free of rank 1 over $\mathbb{I}$ and for each L/K can define

$$\eta_L : H^0(L, \mathbb{D}^\dagger) \longrightarrow \mathbb{D} \twoheadrightarrow \mathbb{J} \xrightarrow{\eta} \mathbb{I}.$$

Definition

Let $G_n := \mathrm{Gal}(K_n/K)$ and $\mathcal{Q}_n := \mathrm{Cor}_{H_{p^{n+1}}/K_n}(\mathcal{P}_{p^{n+1}}) \in H^0(K_n, \mathbb{D}^\dagger)$.
The *n-th big theta element* is

$$\Theta_n := \sum_{\sigma \in G_n} \eta_{K_n}(\mathcal{Q}_n^\sigma) \otimes \sigma^{-1} \in \mathbb{I}[G_n].$$

A conjecture

- By horizontal compatibility, the maps $\mathbb{I}[G_{n+1}] \longrightarrow \mathbb{I}[G_n]$ send
$$\Theta_{n+1} \longmapsto \mathbf{a}_p \cdot \Theta_n.$$
- Define
$$\mathcal{L}_p(\mathbf{f}/K) := \Theta_\infty \cdot \Theta_\infty^*,$$
where $\Theta_\infty := \varprojlim_n \mathbf{a}_p^{-n} \cdot \Theta_n \in \mathbb{I}[[G_\infty]]$.

Let $w = \pm 1$ be the generic root number of f_ν over $\mathbb{Q}$, for $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight and trivial nebentypus.

Conjecture (Longo–Vigni)

(A) *Let ν be a non-exceptional arithmetic prime of even weight $k_\nu \geq 2$. Then for all nontrivial $\chi : G_\infty \longrightarrow \mathbb{C}_p^\times$ of finite order*
$$(\chi \circ \nu)(\mathcal{L}_p(\mathbf{f}/K)) \neq 0 \iff L(f_\nu, \chi, k_\nu/2) \neq 0.$$

(B) *Assume $w = 1$. Then the element $\mathbb{1}_K(\mathcal{L}_p(\mathbf{f}/K)) \in \mathbb{I}$ is nonzero.*

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Automorphic forms on definite quaternion algebras

- A: $\mathbb{Z}_p$-module with right linear action of $M_2(\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Q}_p)$.

Definition

An *A-valued automorphic form on B of level $U \subset \hat{B}^\times$* is a function

$$\phi : \hat{B}^\times \longrightarrow A$$

such that $\phi(gbu) = \phi(b)\imath_p(u_p)$ for all $g \in B^\times$, $b \in \hat{B}^\times$, and $u \in U$.

- For R a $\mathbb{Z}_p$-algebra, let $L_k(R)$ be the module of homogeneous polynomials $P(X, Y) \in R[X, Y]$ of degree $k - 2$ with right action

$$P| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = P(dX - cY, -bX + aY).$$

- Notation: $S_k(U; R) := S(U; L_k(R))$.

Higher weight theta elements

Theorem (Jacquet–Langlands)

There exist Hecke-equivariant isomorphisms JL:

$$S_k(\hat{R}_0) \xrightarrow{\sim} S_k(\Gamma_0(N))^{N^- -\mathrm{new}};$$
$$S_k(U_m) \xrightarrow{\sim} S_k(\Gamma_0(N) \cap \Gamma_1(p^m))^{N^- -\mathrm{new}}.$$

- $f \in S_k(\Gamma_0(N))$ p-ordinary newform.
- $\phi_f = \mathrm{JL}(f) \in S_k(\hat{R}_0)$ *p-adically normalised*:

$$\phi_f \not\equiv 0 \pmod{p}.$$

- Let $\alpha_p \in \overline{\mathbb{Q}}_p^\times$ be the p-adic unit root of $X^2 - a_p(f)X + p^{k-1} = 0$, and consider the *$p$-stabilization*:

$$\tilde{\phi}_f := \phi_f - \frac{p^{k/2-1}}{\alpha_p}\phi_f | \begin{pmatrix} 1 & \\ & p \end{pmatrix}.$$

Higher weight theta elements

- Define $\tilde{\phi}_f^{[r]}$ by

$$\tilde{\phi}_f = \sum_{r=0}^{k-2} \binom{k-2}{r} (-1)^r \tilde{\phi}_f^{[r]} \otimes \mathbf{v}_r : B^\times \backslash \widehat{B}^\times \longrightarrow L_k(\mathcal{O}_F),$$

where $\mathbf{v}_r \longleftrightarrow X^r Y^{k-2-r}$.

Definition (Chida–Hsieh)

Let $\mathcal{G}_{n+1} := \mathrm{Gal}(H_{p^{n+1}}/K)$ and $P_{p^{n+1}} = [1] \in K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_{p^{n+1}} \cong \mathcal{G}_{n+1}$. The *n-th theta element* $\theta_n(f)$ associated to f is the image of

$$\sum_{\sigma \in \mathcal{G}_{n+1}} \tilde{\phi}_f^{[k/2-1]}(P_{p^{n+1}}^\sigma) \otimes \sigma^{-1}$$

under $\mathcal{O}_F[\mathcal{G}_{n+1}] \longrightarrow \mathcal{O}_F[G_n]$.

Gross' special value formula in higher weights

- The natural maps $\mathcal{O}_F[G_{n+1}] \longrightarrow \mathcal{O}_F[G_n]$ send

$$\theta_{n+1}(f) \longmapsto \alpha_p \cdot \theta_n(f).$$

- Define

$$L_p(f/K) := \theta_\infty(f) \cdot \theta_\infty(f)^*,$$

where $\theta_\infty(f) := \varprojlim_n \alpha_p^{-n} \cdot \theta_n(f) \in \mathcal{O}_F[[G_\infty]]$.

Theorem (Chida–Hsieh)

For all $\chi : G_\infty \longrightarrow \mathbb{C}_p^\times$ of finite order,

$$\chi(L_p(f/K)) = (*) \cdot L^{\mathrm{alg}}(f, \chi, k/2),$$

where $L^{\mathrm{alg}}(f, \chi, k/2) := \frac{L(f,\chi,k/2)}{\Omega_{f,N^-}}$, with $\Omega_{f,N^-} \in \mathbb{C}^\times$ Gross' period.

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Higher weight specializations

- If ν is an arithmetic prime of $\mathbb{I}$ of weight $k_\nu \geq 2$ and level $m_\nu \geq 0$, then $f_\nu := \nu(\mathbf{f})$ is the p-stabilization of a newform $f_\nu^\sharp$ of weight k_ν.

Theorem (C.–Longo, in progress)

Let $f \in S_{k_o}(\Gamma_0(N))$ be a p-ordinary newform. Then there exists a constant $C > 0$ such that for all ν of weight $k_\nu \equiv k_o \pmod{2(p-1)p^C}$ and trivial nebentypus,

$$\nu(\Theta_\infty) = \theta_\infty(f_\nu^\sharp)$$

as elements in $\mathcal{O}_\nu[[G_\infty]]$.

Corollary

- *For all ν as in the Theorem, Conjecture A holds.*
- *Conjecture B holds if and only if $L(f_\nu, \mathbb{1}_K, k_\nu/2) \neq 0$ for all but finitely many ν as in the Theorem.*

Sketch of the proof

Rough Idea:

- Relate Θ_∞ to $\mathbf{f}$ using JL in p-adic families.
- More precisely, in view of the identification

$$\mathrm{Pic}(\widetilde{X}_m) \otimes_{\mathbb{Z}} \mathcal{O}_F = \mathcal{O}_F[B^\times \backslash \hat{B}^\times / U_m],$$

one might hope to "evaluate $\mathrm{JL}(\mathbf{f})$ at the big Heegner points $\mathcal{P}_c$".

p-adic JL in p-adic families (pre-Buzzard–Chenevier)

- $\mathbb{D}$: $\mathcal{O}_F$-valued measures on $(\mathbb{Z}_p^2)'$ with natural right $\mathbf{GL}_2(\mathbb{Z}_p)$-action.
- $\mathbb{W} := S(\hat{R}_0; \mathbb{D})$.

Specialization maps: If ν has weight $k_\nu \geq 2$ and level $m_\nu \geq 0$,

$$\begin{aligned} \rho_\nu : \mathbb{D} &\longrightarrow L_{k_\nu}(\mathcal{O}_\nu) \\ \mu &\longmapsto \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \varepsilon_\nu(x)(xY - yX)^{k_\nu - 2} d\mu(x, y). \end{aligned}$$

Theorem (Greenberg–Stevens $+\varepsilon$)

There exists $\Phi \in e^{\mathrm{ord}}\mathbb{W}$ and $C > 0$ such that:

- *For all ν of weight $k_\nu \equiv k_o \pmod{(p-1)p^C}$,*

  $$\rho_{\nu,*}(\Phi) = \lambda_\nu \cdot \tilde{\phi}_{f_\nu}$$

  for some $\lambda_\nu \in \mathbb{C}_p$.
- $\rho_{\nu_o,*}(\Phi) = \tilde{\phi}_f$.

Partial p-adic L-functions

- Decompose $\varepsilon_{\mathrm{cyc}} = \omega \cdot \epsilon_w$, and define
$$\Theta : G_{\mathbb{Q}} \longrightarrow \Lambda^\times := (\mathbb{Z}_p[[1 + p\mathbb{Z}_p]])^\times$$
by $\Theta(\sigma) = \omega^{k_o/2-1}(\sigma) \cdot [\epsilon_w(\sigma)^{1/2}]$.
- Define $\theta : \mathbb{Z}_p^\times \longrightarrow \Lambda^\times$ by
$$\Theta = \theta \circ \varepsilon_{\mathrm{cyc}}.$$
- If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has weight 2, then $\theta_\nu^2 = \varepsilon_\nu$ is the nebentypus of f_ν.

Definition

The *partial p-adic L-function* associated to $\mathbf{f}$ and $P_c = [1] \in \mathrm{Pic}(\mathcal{O}_c)$ is
$$\mathcal{L}_p(\mathbf{f}^\dagger/K, P_c; \nu) := \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \nu(x)\theta_\nu(y/x)d\Phi(P_c)(x,y),$$
seen as a continuous function of $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Weight 2 specializations

Lemma

For all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and wild level $m \geq 2$,

$$\mathcal{L}_p(\mathbf{f}^\dagger/K, P_c; \nu) = \lambda_\nu \cdot \nu(\mathbf{a}_p)^{-m} \cdot (\phi_\nu \otimes \theta_\nu^{-1})(P_c).$$

- Idea: specialize Φ at ν, using the Control Theorem.

Lemma

For all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and wild level $m \geq 2$,

$$\nu(\eta^{\Phi}_{H_c}(\mathcal{P}_c)) = \lambda_\nu \cdot \nu(\mathbf{a}_p)^{-m} \cdot (\phi_\nu \otimes \theta_\nu^{-1})(P_c).$$

where $\eta^{\Phi}_{H_c} : H^0(H_c, \mathbb{D}^\dagger) \longrightarrow \mathbb{D} \longrightarrow \mathbb{J} \xrightarrow{\eta} \mathbb{I}$ *"corresponds" to* Φ.

- Idea: specialize $\mathcal{P}_c$ at ν, tracing through the construction of $\mathcal{P}_c$.

Corollary

$\mathcal{L}_p(\mathbf{f}^\dagger/K, P_c; \nu) = \eta^{\Phi}_{H_c}(\mathcal{P}_c)$ *as continuous functions on* $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

End of proof

- By the construction of theta elements, it suffices to show

$$\nu(\eta^{\Phi}_{H_c}(\mathcal{P}_c)) = \lambda_\nu \cdot \tilde{\phi}^{[k/2-1]}_{f^\sharp_\nu}(P_c).$$

- Consider the continuous function on $\mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ given by

$$\mathcal{L}_p(\mathbf{f}/K, P_c; \nu, \sigma) := \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \nu(x)\sigma(y/x)d\Phi(P_c)(x,y).$$

- For each ν as in the statement of the Theorem,

$$\theta_\nu(z) = \sigma_{k/2-1}(z) = z^{k/2-1}. \qquad (*)$$

Hence, on the one hand

$$\nu(\eta^{\Phi}_{H_c}(\mathcal{P}_c)) \overset{\mathit{Corollary}}{=} \mathcal{L}_p(\mathbf{f}^\dagger/K, P_c; \nu) \overset{(*)}{=} \mathcal{L}_p(\mathbf{f}/K, P_c; \nu, \sigma_{k/2-1}).$$

A calculation from Greenberg–Stevens

- On the other hand, following Greenberg–Stevens:

$$\sum_{r=0}^{k-2}\binom{k-2}{r}(-1)^r\mathcal{L}_p(\mathbf{f}/K,P_c;\nu,\sigma_r)\cdot X^rY^{k-2-r}$$
$$=\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p}\sum_{r=0}^{k-2}\binom{k-2}{r}(-1)^rx^{k-2-r}y^r d\Phi(P_c)(x,y)\cdot X^rY^{k-2-r}$$
$$=\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p}(xY-yX)^{k-2}d\Phi(P_c)(x,y)=\lambda_\nu\cdot\tilde{\phi}_{f_\nu^\sharp}(P_c).$$

- Looking at the coefficient of $X^{k/2-1}Y^{k/2-1}$:

$$\mathcal{L}_p(\mathbf{f}/K,P_c;\nu,\sigma_{k/2-1})=\lambda_\nu\cdot\tilde{\phi}_{f_\nu^\sharp}^{[k/2-1]}(P_c).$$

Final Comments

- The construction of 2-variable p-adic L-functions via big Heegner points can be lifted to localized Hida–Hecke algebras (rather than just a single branch $\mathbb{I}$).
- **Application**: anticyclotomic analogues of the results of Emerton–Pollack–Weston (ongoing joint work with C.-H. Kim and M. Longo).
- Our Theorem yields an interpolation of Gross' special value formula for finite order anticyclotomic twists of the forms f_ν in a Hida family.
- **Hope**: Building on this interpolation (for the twist by $\mathbb{1}_K$), one can make progress on Howard's "horizontal nonvanishing conjecture" under some assumptions on N^- extending Wei Zhang's arguments to certain (non-arithmetic) height one primes of $\mathbb{I}$.

Happy Birthday, Prof. Stevens!