Explicit regulators for Rankin-Selberg products of higher weight modular forms

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Beilinson conjecture

- X : smooth projective variety over $\mathbb{Q}$
- $L(h^i(X), s)$: L-function for $M = h^i(X)$
- $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) := (K_{2j-i}(X) \otimes \mathbb{Q})^{(j)}$: motivic cohomology
  Then, $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \simeq \mathrm{CH}^j(X, 2j-i) \otimes \mathbb{Q}$
- $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}$: integral part (image from integral model)
- $H^i_{\mathcal{D}}(X, \mathbb{R}(j))$: Deligne cohomology
- $r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$: regulator map
- For $i > \frac{j}{2} + 1$, we have an exact sequence

  $$0 \to H^i_{\mathrm{B}}(X_{/\mathbb{R}}, \mathbb{R}(j)) \to H^i_{\mathrm{dR}}(X_{\mathbb{R}})/F^j \to H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(j)) \to 0$$
- Define $\mathcal{D}_{i,j} := \det_{\mathbb{Q}} H^i_{\mathrm{dR}}(X)/F^j \otimes \det_{\mathbb{Q}} H^i_{\mathrm{B}}(X_{/\mathbb{R}}, \mathbb{Q}(j))^{\vee}$

Beilinson conjecture

Beilinson Conjecture:
Assume $n > \frac{i}{2} + 1$. Then

1. $r_{\mathcal{D}} \otimes \mathbb{R} : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes \mathbb{R} \to H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(n))$ is an isomorphism.
2. $r_{\mathcal{D}}(\det_{\mathbb{Q}} H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}) = L(h^i(X), n)\mathcal{D}_{i,n}$ in $\det_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(n))$.

Remark: We can refine the conjecture for pure motives.

Remark: For the application to Tamagawa number conjecture, we need explicit computation of regulators.

Regulators for Rankin-Selberg products

Notations:

- $k, \ell \geq 0,\ N \geq 1$: integers satisfying $k \leq \ell$
- $Y = Y(N)_{/\mathbb{Q}}$: modular curve of full level N-structure
- $\pi : E \to Y$: universal elliptic curve
- $E^k := E \times_Y \cdots \times_Y E \xrightarrow{\pi_k} Y$: k-fold fiber product
- $X = X(N)_{/\mathbb{Q}}$: compactified modular curve
- $X^\infty := X(N) \setminus Y(N)$: cusps
- $\bar{\pi} : \overline{E} \to X$: universal generalized elliptic curve
- $\overline{E}^k := \overline{E} \times_X \cdots \times_X \overline{E} \xrightarrow{\bar{\pi}_k} X$: k-fold fiber product
  (This is not smooth if $k \geq 2$)
- $\hat{E}^k$: Néron model of E^k over X
- $\overline{\overline{E}}^k \to \overline{E}^k$: Deligne's desingularization (smooth projective)

Regulators for Rankin-Selberg products

Beilinson's Eisenstein symbol:

- $\mathrm{Eis}^k : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0 \to H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))^{\varepsilon_k}$
  : Eisenstein symbol
- ε_k : sign character for the symmetric group Σ_{k+1} acting on $E^k \subset E^{k+1}$
- Beilinson calculated the realization of Eisenstein symbol

$$\mathrm{Eis}^k_{\mathcal{D}}(\beta) := r_{\mathcal{D}}(\mathrm{Eis}^k(\beta)) \in H^{k+1}_{\mathcal{D}}(E^k, \mathbb{R}(k+1))$$

  for $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$ and showed that this element is represented by Eisenstein series.

Regulators for Rankin-Selberg products

Fix an integer j satisfying $0 \leq j \leq k \leq \ell$.
Put $k' := k - j$ and $\ell' := \ell - j$.

$$E^k(\mathbb{C}) = \coprod_{(\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathcal{H} \times \mathbb{C}^k/\mathbb{Z}^{2k} \ni [(\tau; z_1, \ldots, z_k)]$$

Consider the following morphisms:

- $p : E^{k'+j+\ell'} \to E^{k'+\ell'}$,
  $(\tau; u, t, v) \mapsto (\tau; u, v)$
- $\Delta : E^{k'+j+\ell'} \to E^{k'+2j+\ell'}$,
  $(\tau; u, t, v) \mapsto (\tau; u, t, t, v)$
- $i : E^{k'+2j+\ell'} = E^{k+\ell} \to E^k \times E^\ell$,
  $(\tau; u, t, t', v) \mapsto ((\tau; u, t), (\tau; t', v))$

Regulators for Rankin-Selberg products

Then we have the following maps:

$$\begin{aligned} & H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+\ell'}, \mathbb{Q}(k'+\ell'+1))^{\varepsilon_{k'+\ell'}} \\ & \xrightarrow{p^*} H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+j+\ell'}, \mathbb{Q}(k'+\ell'+1))^{\varepsilon_{k'+j+\ell'}} \\ & \xrightarrow{\Delta_*} H_{\mathcal{M}}^{k+\ell+1}(E^{k+\ell}, \mathbb{Q}(k+\ell-j+1))^{\varepsilon_{k+\ell}} \\ & \xrightarrow{i_*} H_{\mathcal{M}}^{k+\ell+3}(E^k \times E^\ell, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k, \varepsilon_\ell)} \end{aligned}$$

For $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$, define

$$\mathrm{Eis}^{k,\ell,j}(\beta) := i_* \circ \Delta_* \circ p^*(\mathrm{Eis}^{k'+\ell'}(\beta)),$$

$$\mathrm{Eis}_{\mathcal{D}}^{k,\ell,j}(\beta) := r_{\mathcal{D}}(\mathrm{Eis}^{k,\ell,j}(\beta)).$$

Regulators for Rankin-Selberg products

Setting:

- $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)^{\mathrm{new}}$, $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)^{\mathrm{new}}$: normalized eigenforms
- Put $N := lcm(N_f, N_g)$ and $\chi := \chi_f^{-1} \cdot \chi_g^{-1}$
- $g^*(\tau) := \sum_{n=1}^{\infty} \overline{a_n(g)} e^{2\pi i n \tau}$, where $a_n(g)$ is the n-th Fourier coefficient of g.
- $\omega_f := (2\pi i)^{k+1} f(\tau) d\tau \wedge dz_1 \wedge \cdots \wedge dz_k$
  $\omega_{g^*} := (2\pi i)^{\ell+1} g^*(\tau) d\tau \wedge dz_{k+1} \wedge \cdots \wedge dz_{k+\ell}$

Theorem:

$$\langle \mathrm{Eis}_{\mathcal{D}}^{k,\ell,j}(\beta_\chi), \omega_f \wedge \overline{\omega_{g^*}} \rangle = (2\pi i)^{-(k+\ell-2j)} C_{k,\ell,j} \cdot L(f \otimes g, k+\ell-j+2)$$

Regulators for Rankin-Selberg products

Theorem:

$$\langle \mathrm{Eis}_{\mathcal{D}}^{k,\ell,j}(\beta_\chi), \omega_f \wedge \overline{\omega_{g^*}} \rangle = (2\pi i)^{-(k+\ell-2j)} C_{k,\ell,j} \cdot L(f \otimes g, k+\ell-j+2)$$

where

- $C_{k,\ell,j} = (\pm 1) \times (k+\ell-j+1)! \times \frac{(k-j)!(\ell-j)!}{(k+\ell-2j)!}$
- $(\pm 1) = (-1)^{(k-\ell)(\ell-j+1)+\frac{(\ell-j)(\ell-j+1)}{2}+\frac{(k-j-2)(k-j-1)}{2}+\frac{j(j-1)}{2}}$
- $$\beta_\chi(x,y) = \begin{cases} \frac{1}{N}\sum_{a \in \mathbb{Z}/N\mathbb{Z}} \overline{\chi}(a) e^{\frac{2\pi i a x}{N}} & \text{if } y=0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

  Then $\beta_\chi \in \mathbb{Q}(\zeta_N)[(\mathbb{Z}/N\mathbb{Z})^2]^0$ (need to extend coefficients).

Remark: (i) Similar result is also obtained by recent work of Kings-Loeffler-Zerbes and (unpublished work of) Scholl.
(ii) $k = \ell = 0$: Baba-Sreekantan and Bertolini-Darmon-Rotger.

Residues

Consider the localization sequence for $(\hat{E}^k, E^k)$:

$$0 \to H^{k+1}_{\mathcal{M}}(\hat{E}^k, \mathbb{Q}(k+1))^{\varepsilon_k} \to H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))^{\varepsilon_k} \overset{\mathrm{Res}^k}{\to} \mathcal{F}^k_N \to 0$$

where $\mathcal{F}^k_N \simeq H^0_{\mathcal{M}}(X^\infty, \mathbb{Q}(0)) \simeq \mathbb{Q}[X^\infty]$ is defined by

$$\mathcal{F}^k_N := \left\{ f : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Q} \,\middle|\, f(g \cdot \begin{pmatrix} a & b \\ 0 & \pm 1 \end{pmatrix}) = (\pm 1)^k f(g) \right\}.$$

Define $\omega^k_N : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0 \to \mathcal{F}^k_N$ by

$$\omega^k_N(\beta)(g) = N^k \sum_{x=(x_1,x_2)\in(\mathbb{Z}/N\mathbb{Z})^2} \beta(g \cdot^t x) B_{k+2}(\left\langle \frac{x_2}{N} \right\rangle).$$

__**Theorem**__ (Schappacher-Scholl):

$$\mathrm{Res}^k \circ \mathrm{Eis}^k = C_k \cdot \omega^k_N \text{ with } C_k = \frac{k+1}{(k+2)!}.$$

Residues

$Z^k := \hat{E}^k \setminus E^k = \mathbb{G}_m^k \times_X X^\infty$ ($\hat{E}^k$: Néron model of E^k over X)
$U^{k,\ell} := \hat{E}^k \times \hat{E}^\ell \setminus Z^k \times Z^\ell$
$i' : E^{k+\ell} \to U^{k,\ell}$: canonical closed immersion
$i'_* : H_{\mathcal{M}}^{k+\ell+1}(E^{k+\ell}, \mathbb{Q}(k+\ell-j+1)) \to H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))$

We have a localization sequence:

$$\begin{aligned} H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^k \times \hat{E}^\ell, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)} \\ \longrightarrow H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)} \\ \xrightarrow{\mathrm{Res}^{k,\ell,j}} H_{\mathcal{M}}^{k+\ell}(Z^k \times Z^\ell, \mathbb{Q}(k+\ell-j))^{(\varepsilon_k,\varepsilon_\ell)} \end{aligned}$$

$H_{\mathcal{M}}^{k+\ell}(Z^k \times Z^\ell, \mathbb{Q}(k+\ell))^{(\varepsilon_k,\varepsilon_\ell)}$ can be identified with $\mathcal{F}_N^k \otimes \mathcal{F}_N^\ell$.

Theorem:

1. $\mathrm{Res}^{k,\ell,j} \circ \mathrm{Eis}^{k,\ell,j} = 0$ except for $j = 0$.
2. $\mathrm{Res}^{k,\ell,0} \circ \mathrm{Eis}^{k,\ell,0}(\beta) = C_k \cdot C_\ell \cdot \omega_N^k(\beta) \otimes \omega_N^\ell(\beta)$.

Vertical and horizontal cycles for $j = 0$

$i^1_{\mathrm{cusp}} : Z^k \times E^\ell \hookrightarrow U^{k,\ell}$
$i^2_{\mathrm{cusp}} : E^k \times Z^\ell \hookrightarrow U^{k,\ell}$

$$H^{k+\ell+1}_{\mathcal{M}}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1)) \stackrel{i^1_{\mathrm{cusp},*}}{\to} H^{k+\ell+3}_{\mathcal{M}}(U^{k,\ell}, \mathbb{Q}(k+\ell+2))$$

$$H^{k+\ell+1}_{\mathcal{M}}(E^k \times Z^\ell, \mathbb{Q}(k+\ell+1)) \stackrel{i^2_{\mathrm{cusp},*}}{\to} H^{k+\ell+3}_{\mathcal{M}}(U^{k,\ell}, \mathbb{Q}(k+\ell+2))$$

For $c \in X^\infty$: cusp, $\hat{\pi}_k^{-1}(c) \simeq \mathbb{G}_m^k$. Then we have a canonical generator $\eta_c \in H^k_{\mathcal{M}}(\hat{\pi}_k^{-1}(c), \mathbb{Q}(k)) \simeq \mathbb{Q}$.

Also we have an element $\mathrm{Eis}^\ell(\beta) \in H^{\ell+1}_{\mathcal{M}}(E^\ell, \mathbb{Q}(\ell+1))$. Then,

$$\eta_c \cup \mathrm{Eis}^\ell(\beta) \in H^{k+\ell+1}_{\mathcal{M}}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))$$

and

$$\mathrm{Res}^{k,\ell,0}(i^1_{\mathrm{cusp},*}(\eta_c \cup \mathrm{Eis}^\ell(\beta))) = \{c\} \otimes C_\ell \cdot \omega^\ell_N(\beta).$$

Since ω^ℓ_N is surjective (Beilinson), there exists a linear combination ξ_β of $i^1_{\mathrm{cusp},*}(\eta_{c_1} \cup \mathrm{Eis}^\ell(\beta_1))$ and $i^2_{\mathrm{cusp},*}(\mathrm{Eis}^k(\beta_2) \cup \eta_{c_2})$ such that

$$\mathrm{Res}^{k,\ell,0}(\xi_\beta) = C_k \cdot C_\ell \cdot \omega^k_N(\beta) \otimes \omega^\ell_N(\beta).$$

Generalized Beilinson-Flach elements

Define the generalized Beilinson-Flach element by

$$\mathrm{BF}^{k,\ell,j}(\beta) := \begin{cases} \mathrm{Eis}^{k,\ell,j}(\beta) & \text{if } j \neq 0, \\ \mathrm{Eis}^{k,\ell,0}(\beta) - \xi_\beta & \text{if } j = 0. \end{cases}$$

Theorem:

$$\begin{aligned} \mathrm{BF}^{k,\ell,j}(\beta) \in \mathrm{Im}\,[H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^k \times \hat{E}^\ell, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)} \\ \to H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)}] \end{aligned}$$

Remark: For Beilinson conjecture, we still need to show

$$\begin{aligned} H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^k \times \hat{E}^\ell, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)} \\ \simeq H_{\mathcal{M}}^{k+\ell+3}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(k+\ell-j+2))^{(\varepsilon_k,\varepsilon_\ell)} \end{aligned}$$

and the integrality of the elements.