6-3-2014 Samit Dasgupta Glennfest 16:45pm.

The Eisenstein cocycles and Gross's Tower of Fields conjecture

jt. with Michael Spiess.

F, tot. real field.

$$\begin{array}{ccc} & & L \\ \tilde{G} & & \Big| H \\ & & K \\ & & \Big| G \\ & & F \end{array}$$

$\tilde{G}$, finite abelian

S = fin. set of places of F

$\supset S_\infty \cup S_{ram}(L/F)$

$S \cap T = \phi$

T $\to$ smoothing operation

T = finite set of places satisfying some condition

$$I = \mathrm{Ker}(\mathbb{Z}[\tilde{G}] \to \mathbb{Z}[G])$$

For $\sigma \in \tilde{G}$,

$$\zeta_S(\sigma, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ (\mathfrak{a}, S) = 1 \\ \mathrm{Frob}(L/F, \mathfrak{a}) = \sigma}} \frac{1}{N\mathfrak{a}^s}$$

$$\Theta_{S,T}^{L/F}(s) = \left(\sum_{\sigma \in \tilde{G}} \zeta_S(\sigma, s)[\sigma^{-1}] \right) \cdot \left(\prod_{\mathfrak{p} \in T} \left(1 - N\mathfrak{p}^{1-s}[\mathrm{Frob}(L/F, \mathfrak{p})^{-1}]\right) \right)$$

$\in \mathbb{C}[G]$

Thm (Casson-Nogues 1979, Deligne-Ribet 1980)

$$\theta_{S,T}^{L/F}(0) \in \mathbb{Z}[\tilde{G}].$$

Conj. (Gross, 1988)

Let r = # places in S that splits completely in K

$$\theta_{S,T}^{L/F}(0) \in I^r.$$

Thm (Popescu-Greither 2012).

Let K be totally complex.

p-part is true. Let p be an odd prime.

Assume $S \supset S_p$.

Then

$$\theta_{S,T}^{L/F}(0) \in I^r \otimes \mathbb{Z}_p \subset \mathbb{Z}_p[\hat{G}]$$

Thm (Dasgupta-Spiess)

Let K be totally complex.

Then Gross conj. is true.

Shintani

Shintani's method.

Ex. K, narrow Hilbert class field of F.

Given $\sigma \in G = \mathrm{Gal}(K/F)$,

Fix $\mathfrak{f}$ s.t. $(\mathfrak{f}, S) = 1$

and $\mathrm{Frob}(L/F, \mathfrak{f}) = \sigma^{-1}$.

$$\zeta_S(\sigma, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ \mathfrak{a} \sim \mathfrak{f}^{-1} \\ (\mathfrak{a}, S) = 1}} \frac{1}{N\mathfrak{a}^s}$$

$$\mathfrak{a} \cdot \mathfrak{f} = (\alpha) \; ; \; \underline{\underline{\alpha \gg 0}}$$

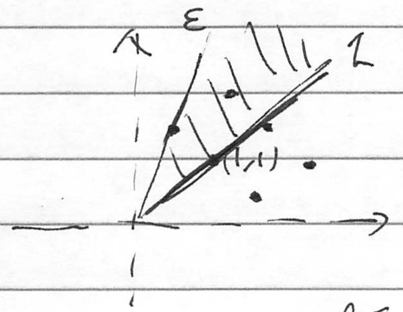
narrow.

$$= N\mathfrak{f}^s \cdot \sum_{\substack{\alpha \in \mathfrak{f} / E^+ \\ \alpha \gg 0 \\ (\alpha, S) = 1}} \frac{1}{N\alpha^s}.$$

tot. pos unit.

Embed $F \hookrightarrow \mathbb{R}^n$ where $n = [F : \mathbb{Q}]$.

$\underline{n=2}$, $\mathcal{D}$: fund. domain.



$$= N\mathfrak{f}^s \cdot \sum_{\substack{\alpha \in \mathfrak{f} \cap \mathcal{D} \\ (\alpha, S) = 1}} \frac{1}{N\alpha^s}$$

Eisenstein cocycle

* Let $N = \{ \text{functions } (\mathbb{R}^{>0})^n \to \mathbb{Z} \}$

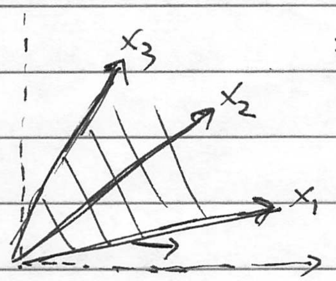
$$\begin{array}{c} \circlearrowleft \\ (\mathbb{R}^{>0})^n \quad \text{gp action.} \\ \cup \\ F_+^\times . \end{array}$$

For $x_1, \cdots, x_n \in (\mathbb{R}_{>0})^n$,

$$C(x_1, \cdots, x_n) = \{ \sum t_i \cdot x_i : t_i \in \mathbb{R}_{>0} \}.$$

$$\Phi_0(x_1, \cdots, x_n) = \mathrm{sgn}(\det(x)) \cdot \mathbb{1}_{C^*(x_1, \cdots, x_n)}$$

$$= \mathrm{sgn}(\det(x)) \cdot \lim_{\varepsilon \to 0} \mathbb{1}_{C(x_1, \cdots, x_n)} (x + (\varepsilon, 0, \cdots, 0))$$



$$[\Phi_0] \in H^{n-1}(F_+^\times, N).$$

* Let $\mathcal{U} \subset \mathbb{A}_F^f$ = finite adeles

$$\zeta(x_1, \cdots, x_n; \mathcal{U}, s) = \sum_{\substack{\alpha \in C^*(x_1, \cdots, x_n) \\ \alpha \in F \cap \mathcal{U}}} \frac{1}{N_m \alpha^s}$$

Define

$$\Phi(x_1,\cdots,x_n)(\mathcal{U})$$

$$:= \mathrm{sgn}(\det(x_1)) \cdot \zeta(x_1,\cdots,x_n,\mathcal{U},0)$$

$$\underline{\underline{\in \mathbb{Q}.}}$$

Smooth using primes in T

"Casson–Nogues trick"

$\rightsquigarrow [\Phi_T] \in H^{n-1}(F_T^\times, \mathrm{Meas}(\mathbb{A}_F, \mathbb{Z}))$

$F_T^\times \subset F_+^\times$

$\uparrow$

elts with ord=0 at all primes in T.

R, comm. ring with discrete top.

$\mathcal{C}_c(\mathbb{A}_F, R) = \{$ cptly supported conti. fhn on $\mathbb{A}_F$ valued in $R\}$

$$\mathrm{Meas}(\mathbb{A}, \mathbb{Z}) \times \mathcal{C}_c(\mathbb{A}, R) \to R.$$
$$(\mu, f) \longmapsto \int f d\mu$$

Induces

$$H^{n-1}(F_T^\times, \mathrm{Meas}(\mathbb{A}_F, \mathbb{Z})) \times H_{n-1}(F_T^\times, \mathcal{C}_c(\mathbb{A}_F R)) \to R$$
$$(\Phi_T, \rho) \mapsto \Phi_T \cap \rho$$

We define

$$\rho_{L/F} \in H_{n-1}(F_T^\times, \mathcal{C}_c(\mathbb{A}_F, \mathbb{Z}[\tilde{G}]))$$

corresponds to

① taking a generator of $H_{n-1}(E_+, \mathbb{Z})$

② mapping $\alpha \in \mathbb{A}_F^\times$

to $\mathrm{rec}_{L/F}(\alpha) \in \tilde{G}$.

Thm 1. $\Theta^{L/F}_{S,T}(0) = \Sigma_T \cap \rho_{L/F} \in \mathbb{Z}[G]$

Gross conj. $\quad X = \{p_1, \dots, p_r\} \subset S$

$\uparrow$ primes splits in K.

Idea: $\rho_{L/F} \equiv 0 \pmod{I^r}$

Thm 2. $\rho_{L/F} \equiv (c_{p_1} \cup \dots \cup c_{p_r}) \cap \rho_{K/F} \mod I^{r+1}$.

- $\rho_{K/F, X} \in H_{n-1+r}(F_T^\times, \mathcal{C}_c(\mathbb{A}_F^{(X)}, \mathbb{Z}[G]))$

$$\begin{tikzcd} L \arrow[d, dash, "H"] \\ K \arrow[d, dash, "G"] \\ F \arrow[uu, bend left, dash, "\tilde{G}"] \end{tikzcd}$$

- $c_p \in H^1(F_p^\times, \mathcal{C}_c(F_p, H))$ if $p \in X$.

$$c_p(x) = (1-x) \cdot (\mathbb{1}_{\mathcal{O}_p} \cdot \mathrm{rec}_p)$$

$$
\begin{tikzcd}
\overbrace{H \times H \times \cdots \times H}^{r} \times \mathbb{Z}[G] \arrow[r] & \overbrace{I/I^2 \times \cdots \times I/I^2}^{r} \times \mathbb{Z}[\tilde{G}]/I \arrow[d, "\text{mult.}"] \\
 & \mathbb{Z}[\tilde{G}]/I^{r+1}
\end{tikzcd}
$$

$H \subset H \otimes \mathbb{Z}[G] \simeq I/I^2.$

RHS clearly $\in I^r$.

Thm 1 + Thm 2 $\Rightarrow$ Gross conj.