6-3-2014 Ellen Eischen Glennfest 9:30 AM.

Eisenstein series, differential operators, and p-adic families of modular forms

p-adic families of Eisenstein series and applications

<u>Goal</u> (for talk):

Discuss recent work p-adically interpolating certain values of particular Eisenstein series on <u>unitary groups</u>

(sign (n,n) including $\mathcal{C}^\infty$)

$E_{\lambda}(A)$ ordinary CM pts. (fixed) vary weight.

<u>Motivation</u> (from L-functions)

(1)

<u>Thm</u> (Kummer, 1800s)

If $k, k' \in 2\mathbb{Z}_{>0}$, s.t. $p-1 \nmid k, k'$

and $k \equiv k' \pmod{(p-1)\cdot p^d}$,

then $\zeta^*(1-k) \equiv \zeta^*(1-k') \pmod{p^{d+1}}$

where $\zeta^*(1-k) = (1-p^{k-1})\cdot\zeta(1-k)$

$\rightsquigarrow$ Kubota–Leopoldt (1960s)
p-adic Dirichlet L-functions

② Katz (1970's)

p-adic L-functions for CM fields

$(L(s, \chi))$

$\uparrow$ Hecke char. on CM field.

Q : For which L-functions, can we construct p-adic analogue, i.e.

Want

$$\text{"}(*) \cdot \mathcal{L}_p(n, X) = (*') \cdot L(n, \underset{\sim}{X})$$

$\uparrow$ p-adic period $\quad$ $\uparrow$ complex period $\quad$ $\uparrow$ can be a family of automorphic forms..

For certain X, (Hecke chars of CM fields, autom. forms on unitary gps, ⋯),

Properties of L-functions $\xleftarrow[\text{doubling}]{\text{Rankin-Selberg}}$ Properties of certain Eisenstein series

The idea

① Choose "nice" Eisenstein series. $(\mathcal{C}^\infty)$

② Compute Fourier coefficients
(where holomorphic)

Katz-Oda
Maass-Shimura operators

③ Apply weight-raising $\mathcal{C}^\infty$ (and p-adic) differential operators to obtain $\mathcal{C}^\infty$ (and p-adic) autom. forms

<u>Note</u>: $(*)\cdot D_\infty E(A) = (*)' \cdot D_{p\text{-adic}} E(A)$

algebraic (for E); ordinary CM pts $/R \hookrightarrow \mathbb{C}$, $\hookrightarrow \mathbb{C}_p$ (for A)

④ Use q-expansion principle and p-adic interpolation of q-expansion coefficients to construct p-adic families.

Setup

- V = n-dim'l v. sp. / CM field $K \supseteq \mathcal{O}_K$.
$$\begin{array}{c} K \\ | \\ E \end{array}$$ E tot. real

(<u>simplification</u> : $E = \mathbb{Q}$)

Fix $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$.

- Fix prime p splits in K.

- $\langle \ , \ \rangle_V$ = nondeg. Herm. pairing on V.

- $W = V \times V$

$\langle \ , \ \rangle_{W} : W \times W \longrightarrow K$.

defined by

$\langle (u, v), (u', v') \rangle = \langle u, u' \rangle_V - \langle v, v' \rangle_V$.

$\langle \ , \ \rangle$ is signature (n, n)

$\rightsquigarrow$ ~~$U(n,n)$~~

$$\underset{\text{sign }(a,b)}{U(V, \langle , \rangle_V)} \times \underset{\text{sign }(b,a)}{U(V, -\langle , \rangle_V)} \hookrightarrow \underset{\text{sign}(n,n)}{\overset{G}{\overset{\shortparallel}{U(W, \langle , \rangle)}}}$$

- P = Siegel parabolic $\quad \chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ Hecke char.

- $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} (\chi \cdot |\ |^{-s}) \quad (s \in \mathbb{C})$.

$\chi \cdot |\ |$: $\chi \circ \det(\ )$, $\det(\ )$

$$E_f(g) := \sum_{\gamma \in P(E) \backslash G(E)} f(\gamma g)$$

- Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

- Fix $k, \nu \in \mathbb{Z}$. $k \geq n$.

$\rightsquigarrow G_{k,\nu,F} = (*) \cdot E_{f(k,\nu,F)}$

p-adic function.

Eischen.

Prop. Let R be $\mathcal{O}_K$-alg. ; $k \geq n$.
Let

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \longrightarrow R$$

be a locally constant function.
supported on

$$(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times GL_n(\mathcal{O}_E \otimes \mathbb{Z}_p).$$

satisfying

$$F(e \cdot x, Nm_{K/E}(e)^{-1} \cdot y) = Nm_{k,\nu}(e) \cdot F(x,y)$$

where $Nm_{k,\nu} = \sigma^{k+2\nu}(\sigma \cdot \bar{\sigma})^{-\nu}$

for all $e \in \mathcal{O}_K^\times$
$x \in \mathcal{O}_K \otimes \mathbb{Z}_p$
$y \in M_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Then $\exists$ automorphic form

$G_{k,\nu,F}$ (on $U(n,n)$)

of wt (k,ν)

defined over R

whose q-expansion at cusp $m \in \underset{(\text{Levi})}{\underline{GM_+}}$

is of form

semicuspform.

$$\sum_{0<\beta\in \underbrace{L_m}_{\text{lattice}}} C(\beta)\cdot q^{\beta}.$$

where $C(\beta)$ is a finite $\mathbb{Z}$-linear combination of terms of form

$$F(a, Nm_{K/E}(a)^{-1}\cdot\beta)\cdot Nm_{k,\nu}(a^{-1}\cdot\det\beta)\cdot Nm_{E/\mathbb{Q}}(\det\beta)^{-n}.$$

already p-depleted

$\Downarrow$

no constant term.

When $R=\mathbb{C}$, this is Fourier expansion at $s=k/2$ of $\mathcal{C}^\infty$-automorphic form

$G_{k,\nu,F}(\cdot, s)$, holom. at $s=k/2$.

Remark. Using p-adic q-expansion principle and p-adic interpolating q-expansion coeffs, can put $G_{k,\nu,F}$ into p-adic family.

Choice of Siegel function f.

$\mathrm{Ind}_P^G(\chi|\cdot|^s)$

Choose $f = \otimes_v f_v$

($\Rightarrow$ Fourier coeff. of E_f factor over v)

(work locally).

$v\mid\infty$: canonical automorphy factors

$$\det(cz+d)^{k+\nu}\cdot\det(\bar{c}({}^t\bar{z})+\bar{d})^{-\nu}.$$

$\rightsquigarrow$ Fourier coeff is $\widehat{\det}(\beta)^{k-n}$.

Shimura

$v\nmid p\cdot\infty$: build out characteristic functions of certain lattices (Shimura)

$$* \text{ Fourier coeff } (*)\cdot\prod_{v\nmid p\infty} P_{\beta,\nu,v}(\chi_E(\pi_v)\cdot|\pi_v|_v^k)$$

poly.

$v \nmid p$:

Given $\tilde{F}$ subject to certain conditions

$$\exists f_{\tilde{F}} \in \bigotimes_{v \nmid p} \mathrm{Ind}_P^G (\chi \cdot | \ |^{2s})$$

whose local Fourier coeff is

$$\tilde{F}(1, {}^t\beta).$$

(Use partial Fourier transform?