6-4-2014 David Hansen Glennfest 9:30 AM.

Iwasawa theory over the eigencurve, and consequences.

	Classical L-functions	p-adic L-functions
They are naturally associated with...	Autom. rep'ns or motives	Same (classical alg. autom. rep'n) + some data "at p" (refinement.
"Naive" def'n	Yes by Euler product	None
Domain	$s \in \mathbb{C}$	$\psi \in \mathrm{Hom}_{cts}(\mathrm{Gal}(F_\infty/F), \overline{\mathbb{Q}}_p^\times)$ $F_\infty \,/\, F$) abelian p-adic Lie ext'n. F : # field
characteristic properties	analytic/meromorphic conti. f.e. good growth (in vertical strips) for converse thm.	same (analytic/merom. conti. f.e.) on $\overline{\mathbb{Q}}_p$-pts of a rigid analytic space. + $L_p(M)(\psi)$ should interpolate $L(0, M \otimes \psi)$ for certain chars ψ. + they vary in p-adic families.
arithmetic meaning at integers (special values and/or order of vanishing)	Conjecturally, related to generalized class gps and/or groups of alg. cycles. Dirichlet, B-SD, Bloch-Kato, Beilinson.	Same, but sometimes (with p-part of the gp.) provably so.

arithmetic meaning.

$\Uparrow$ implies

cohomological interpretation of (all) their zeros

Deninger cohom ??

$\mathbb{F}_1$ - theory ??????

Iwasawa main conjectures

proved in a number of instances

2 Examples.

Ex. χ an even Dirichlet char. of $(\overset{N}{\overset{\shortparallel}{\text{cond}}}, p) = 1$.

$\exists L_p(n, \chi)$ s.t. $L_p(n, \chi) = L_p(n, \chi\omega^{n-1}) \cdot (1 - p^{-n}(\chi\omega)^{n+1}(p))$

analytic

$\forall n \in \mathbb{Z}_{\leq 0}$.

IMC. (Mazur-Wiles)

$L_p(n, \chi)$ controls the χ-part of $Cl(\mathbb{Q}(\zeta_{p^j \cdot N})) \otimes \mathbb{Z}_p \quad \forall j$.

Ex. $f \in S_k(\Gamma_1(N)) \quad p \nmid N$.

Shimura : $\exists \Omega_f^{\pm} \in \mathbb{C}^\times$ s.t. $\forall$ Dirichlet characters η and all $0 \leq j \leq k-2$,

$$L^{alg}(j+1, f \otimes \eta) = \frac{j!\, L(j+1, f\otimes\eta)}{(2\pi i)^{j+1} \cdot \tau(\eta) \cdot \Omega_f^{\pm}} \in \mathbb{Q}(f, \eta)$$

s.t. $(-1)^j = \pm\eta(-1)$

($\Omega_f^{\pm}$ is indep of η.)

$$\mathcal{X} = \mathrm{Spf}\left(\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]\!]\right)^{\mathrm{rig}}.$$

In this case, after choosing a root α of $X^2 - a_f(p)\cdot X + p^{k-1}\cdot \varepsilon_f(p)$.

Can construct fns

$$L_p^{\bullet}(f,\alpha) \in \mathcal{O}(\mathcal{X}) \otimes \mathbb{Q}(f,\alpha)$$

such that

- $L_p^{\bullet}(f,\alpha)(x^j\cdot \underset{\substack{\uparrow \\ \text{cond } p^n \\ n\geq 1}}{\eta(x)}) = \left(\frac{p^{j+1}}{\alpha}\right)^n \cdot L^{\mathrm{alg}}(j+1, f\otimes\eta^{-1})$ $\quad \forall\, 0 \leq j \leq k-2.$

- $L_p^{\bullet}(f,\alpha)$ "has growth of order $\leq v_p(\alpha)$" in a certain sense.

- $\bullet \in \{\mathrm{an}, \mathrm{alg}\}$.

$L_p^{\mathrm{an}}(f,\alpha)$ was constructed by Mazur–Swinnerton-Dyer $(v_p(\alpha)=0)$

Amice–Velu, Vishik $\big)$ $(v_p(\alpha) < k-1)$

Pollack–Stevens, Bellaïche $\big)$ $(v_p(\alpha) = \underset{\substack{\uparrow \\ \text{critical slope.}}}{k-1})$

. $H^{\bullet}$(modular curve)
. autom. forms
⋮

$L_p^{alg}(f, \alpha)$ was constructed by Kato,

using ideas of Perrin-Riou.

- p-adic HT.
- Galois cohomology
  ⋮

order of growth.

Thm (Vishik)

$$L_p^{alg} = L_p^{an} \quad \text{if} \quad v_p(\alpha) < k-1$$

(abstract interpolation property)

This technique does not apply if $v_p(\alpha) = k-1$.

(What about the critical slope case?)

Thm $L_p^{alg} = L_p^{an}$ if $v_p(\alpha) = k-1$ and $\rho_f|_{G_{\mathbb{Q}_p}}$ is indecomposable

proj to wt. is étale.

(decomposable ⇐ Lei-Loeffler-Zerbes)

Still both sides are in the analytic side in the IMC.

Idea of proof:

The data (f, α) determines a pt

$x_{f,\alpha} \in \mathcal{C}_N$ eigencurve.

Bellaïche:

L_p^{an} interpolates into a 2-variable function

can we make it locally integral??

$\mathbb{L}_p^{a} \in \mathcal{O}(\mathcal{X} \times \mathcal{U})$, $\mathcal{U} \subset \mathcal{C}_N$ an affinoid containing $x_{f,\alpha}$.

- $\mathcal{U} \ni \{x_{f',\alpha'}\}$; $\mathbb{L}_p^{an}(x_{f',\alpha'}) = L_p^{an}(f', \alpha')$

many f', α' of varying wt.

$\mathbb{L}_p^{an}(X)$ has growth of order $\leq v_p(U_x)$ $\forall x \in \mathcal{U}(\overline{\mathbb{Q}}_p)$

$U \in \mathcal{O}(\mathcal{C}_N)$ global U_p-eigenvalue.

<u>Suppose</u> $\exists \mathbb{L}_p^{alg}$, i.e. a 2-variable interpolation of Kato's construction

<u>pts with $v_p(\alpha') < k'-1$</u> are dense in $\mathcal{U}$,

$= S$

and $\mathbb{L}_p^{alg} - \mathbb{L}_p^{an}$ vanishes at $S \times \mathcal{X}$ dense in $\mathcal{U} \times \mathcal{X}$

so $\mathbb{L}_p^{alg} - \mathbb{L}_p^{an} = 0$, Now specialize at $x_{f,\alpha}$.

Idea to construct $\mathbb{L}_p^{alg}$ interpolating Kato's construction over the eigencurve.

regulator map

$$\mathrm{Ch}_{L,M}(k, r, k-1) : \varprojlim_n K_2(Y(p^n, M, p^n))$$

$(L+M \geq 5)$

$$\downarrow$$

$$H^1(G_{\mathbb{Q}}, H^1_{\acute{e}t}(Y(LM)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}(T_p E))(2-r))$$

theme! Interpolate this map as k varies.

Thm $(p|M)$. For any affinoid Ω in the wt sp.,

$$\exists\, \mathrm{Ch}_{L,M}(\Omega, r) : \varprojlim_n K_2(\text{same})$$

$$\downarrow$$

Banach sp.

$$H^1(G_{\mathbb{Q}}, H^1_{\acute{e}t}(Y(L,M), \mathbb{D}_{\Omega})(2-r))$$

interpolating $\mathrm{Ch}_{L,M}(k, r, k-1)$.

possible to make it integral. to take $\varprojlim$ (not inverting p!)

quick construction.

$$
\begin{array}{c}
\varprojlim_n K_2(Y_n) \\
\swarrow \\
\varprojlim H^2(Y_n, \mathbb{Z}/p^n(2)) \xrightarrow{\cup x^{k} \zeta_{p^n}^{-r}} \in H^0\left(Y_n, \dfrac{\mathrm{Sym}^{k-2}(T_{a_p}E)(\ )}{p^n}\right)
\end{array}
$$