6-3-2014 Masato Kurihara Glennfest 14:30 pm

Arithmetic of zeta elements
and Rubin-Stark elements.
(jt. with D. Burns and T. Sano.)

- $[K:\mathbb{Q}]=2$.

K img quad, $\#Cl(K) = L(0,\chi)\cdot\frac{w_K}{2}$.

K real, $\#Cl(K) = [\mathcal{O}_K^\times/\{\pm1\} : \text{cyclo. unit}]$

- $K = \mathbb{Q}(\xi_p)$

$A_K - Cl_K \otimes \mathbb{Z}_p$.

$\theta_{K/\mathbb{Q}}$: Stickelberger: rk 0 E.S.

$$\#A_K^\chi = \begin{cases} \#\mathbb{Z}_p/L(0,\chi^{-1}) & \text{if } \chi : \text{odd} \neq \omega \\ [(\mathcal{O}_K^\times)^\chi : \langle c_K^\chi \rangle] & \text{if } \chi : \text{even} \neq 1 \end{cases}$$

K
|) abelian
k : # field.

where $c_K = 1-\xi_p$ ← Euler system. rk 1

Applications for $K/L/k$. including 2 part.

1. Darmon's conj. on the class gp. of real quadratic field.
2. Mazur-Rubin's refined class number formula.

↑ assuming ETNC.

Assuming ETNC,

~~Assuming E~~

# of primes splits completely in S

$$\begin{array}{lll} K & \varepsilon_{K/k,S} \in \bigwedge^{r}(\text{unit}) & \text{rank } r \text{ Euler system} \\ | & & \\ L & \varepsilon_{L/k,S} \in \bigwedge^{r'}(\text{unit}) & \text{rank } r' \text{ Euler system} \\ | & & \\ k & & \end{array}$$

$r' \geq r$ $\quad$ $\ominus \rightsquigarrow$ E.S ? $\quad$ If $r' > r$, $\mathrm{Nm}(\varepsilon_{K/k,S}) = 0$

but still a relation exists

$>$ $\rightsquigarrow$ $\mathrm{Nm}(\varepsilon_{K/k,S}) \equiv \mathrm{Rec}_{L/K}(\varepsilon_{L/k,S})$ mod some ideal.

(Mazur Rubin)

3. Gross's conj. for tori.

$$\theta_{K/k,S}(0)^{*} = 2^{\infty} \frac{h_{L,S}}{h_{k,S}} \cdot R_S.$$

K/k, fin. abel. ext'n of number fields

S, finite set of primes $\supset S_\infty \cup S_{ram}(K/k)$

T, another finite set of primes

s.t. $S \cap T = \phi$

and

$\mathcal{O}^{\times}_{K,S,T}$ is $\mathbb{Z}$-torsion free

$\{ x \in K^{\times} : \mathrm{ord}_v(x) \geq 0$ for all $v \notin S_K$

$x \equiv 1 \pmod{\mathfrak{t}}$ for $\mathfrak{t} \in T_K \}$.

$V \subsetneq S$, each $v \in V$ splits completely in K.
$r = \# V$.

$$\theta_{K/k,S,T}(s) = \prod_{v \in T} \left(1 - \mathrm{Frob}_v^{-1} (Nv)^{1-s}\right)$$

$$\cdot \sum_{\sigma \in G} \zeta_{k,S}(s,\sigma)\,\sigma^{-1}$$

$\theta_{K/k,S,T}(0) \in \underline{\mathbb{Z}[G]}$ 4.

$$\operatorname*{ord}_{s=0} \theta_{K/k,S,T}(s) \geq r$$

$$\theta^{(r)}_{K/k,S,T} := \lim_{s \to 0} \frac{1}{s^r}\,\theta_{K/k,S,T}(s)$$

$$\lambda_{K,S} : \mathcal{O}^{\times}_{K,S} \otimes \mathbb{R} \xrightarrow{\simeq} X_{K,S} \otimes \mathbb{R}$$

$$x \longmapsto -\sum_{w \in S_K} \log |x|_w \cdot w$$

$$\begin{array}{cccc} K & w_1 & w_r & w_0 \\ | & | & | & | \\ V = \{ v_1, & \ldots, & v_r, & v_0 \end{array}$$

$$\varepsilon^{V}_{K/k,S,T} \longmapsto \theta^{(r)}_{K/k,S,T} \bigwedge_{v \in V} (w - w_0)$$

M, $\mathbb{Z}[G]$-module.

$\varphi : \mathrm{Hom}_G(M, \mathbb{Z}[G])$ induces

$$\bigwedge^r M \xrightarrow{\varphi} \bigwedge^{r-1} M.$$

$$m_1 \wedge \cdots \wedge m_r \longmapsto \sum (-1)^{i-1} \varphi(m_i) \cdot m_1 \wedge \cdots \wedge \check{m_i} \wedge \cdots m_r$$

For $\varphi_1, \cdots, \varphi_j \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$,

$$\varphi_1 \wedge \cdots \wedge \varphi_j : \bigwedge^r M \xrightarrow{\varphi_1} \bigwedge^{r-1} M \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_j} \bigwedge^{r-j} M.$$

For a G-module M, $\mathbb{Z}$-torsion free

lattice $\quad \bigcap_G^r M = \{ x \in \bigwedge^r M \otimes \mathbb{Q} : \Phi(x) \in \mathbb{Z}[G]$ for all $\Phi \in \bigwedge^r \mathrm{Hom}(M, \mathbb{Z}[G]) \}$

$\uparrow$ not surjective

$\bigwedge^r M$

<u>Conj.</u> (Rubin) $\varepsilon^V_{K/k,S,T} \in \bigcap^r \mathcal{O}^\times_{K,S,T}$.

$\quad \quad \quad \quad \quad \quad \quad \varepsilon$

Kurihara $\quad \Phi_{K/k}(\varepsilon$

<u>Question</u> $\{ \Phi(\varepsilon) : \Phi \in \bigwedge^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G]) \} \subseteq \mathbb{Z}[G]$

What is the arithmetic meaning of this ideal?

canonical $\mathbb{Z}$ structure of Selmer modules

$S, T.$

$$\begin{array}{c} K \\ | \\ k \end{array}$$

$$\partial_{S,T} : \prod_{w \notin S_K \cup T_K} \mathbb{Z} \longrightarrow \mathrm{Hom}(K_T^\times, \mathbb{Z})$$

$$(n_w) \longmapsto \Big(x \longmapsto \sum_w \mathrm{ord}_w(x) \cdot n_w \Big)$$

$$\mathcal{S}^D_{S,T}(K) = \mathrm{Coker}(\partial_{S,T})$$

$$0 \to Cl^T_S(K)^\vee \to \mathcal{S}^D_{S,T}(K) \to \mathrm{Hom}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}) \to 0.$$

ray class group mod T of $\mathcal{O}_{K,S}$.

$$\mathcal{S}^D_{S,T}(K) = H^2_{c,T}((\mathcal{O}_{K,S})_W, \mathbb{Z})$$

$\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$ induced by $\sigma \mapsto \sigma^{-1}$

$x \longmapsto x^\#$ on G.

inverse action.

Thm 1. Assume ETNC.

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}^D_{S,T}) = \Theta^\#_{K/k,\varepsilon}$$

$$\mathcal{S}_{S,T}(K) := H^{-1}(\underbrace{R\mathrm{Hom}(R\Gamma_c((\mathcal{O}_{K,S})_W, \mathbb{Z}), \mathbb{Z})}_{= C^\bullet})$$

$$0 \to Cl^T_{K,S}(K) \to \mathcal{S}_{S,T}(K) \to X_{K,S} \to 0 \quad \text{exact}$$

where $X_{K,S} = \ker\left(\bigoplus_{v\in S_K} \mathbb{Z} \to \mathbb{Z}\right)$.

Thm 1' $\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{C}l_{S,T}(K)) = \mathcal{I}_{K/k,\varepsilon}$.

Cor assuming ETNC. For any $v \in S\setminus V$,

$$\Phi(\varepsilon) \in \mathrm{Ann}\left(\mathcal{C}l^{T}_{V\cup\{v\}}(K)\right)$$

Ex.

Stickelberger.

$r=0$. $\theta_{K/k,S,T} \in \mathrm{Ann}\left(\mathcal{C}l^{T}_{\emptyset}(K)\right)$

$\theta_{K/k,S,T} \in \mathrm{Fitt}\left(\left(\mathcal{C}l^{T}(K)^{\vee}\right)_p^{-}\right)$ p odd.

$\mathrm{Fitt}_r(M) \subset \mathrm{Fitt}_{r+1}$.

$$\mathbb{Z}[G]^m \xrightarrow{A} \mathbb{Z}[G]^n \to M \to 0$$

~~ideal gen by $(n-r)\times(n-r)$ minors.~~

In our case, $\begin{pmatrix} 0 \\ \hline \\ \end{pmatrix}\}r$

~~$\mathrm{Fitt}_0 = \cdots = \mathrm{Fitt}_{r-1} = 0.$~~

Remark

also can get information on

$\mathrm{Fitt}_{r+i}(Cl_S^T(K))$, $i \in \mathbb{Z}_{\geq 0}$ higher Fitting ideal

$\boxed{\text{Relation with the zeta element}}$

$b_1, \dots, b_d$: basis $(d > r)$ $\longmapsto$ $\in \mathbb{Z}[G]^r$

$$0 \to \mathcal{O}_{K,S,T}^\times \to \underline{P \xrightarrow{\psi} F} \to \mathcal{S}_{S,T} \to 0$$

free $\mathbb{Z}[G]$-module

projective.

$X_{K,S}$ $\cdot$ $w_i - w_0$ for $i = 1, \dots, r$

$\downarrow$

$Y_{K,V}$

$\mathcal{C}^\bullet$

$\mathrm{R}\Gamma$

$\mathcal{C}^\bullet$

$$\vartheta_{K,S} : \det \mathcal{C}^\bullet \otimes \mathbb{R} \xrightarrow{\simeq} \mathbb{R}[G] \quad \bigoplus_{v \in V_K} \mathbb{Z}. \cong \mathbb{Z}[G]^r$$

$$z = \underbrace{z_{K/k,S,T}}_{\text{zeta elt.}} \longmapsto \theta^*_{K/k,S,T}(0) = \sum_\chi L^*_{k,S,T}(\chi, 0)$$

leading term

ETNC :

$$\det \mathcal{C}^\bullet = z \cdot \mathbb{Z}[G].$$

dual base of b_i

$$\psi_i = b_i^* \circ \psi$$

$$\det \mathcal{C}^\bullet \cong \det P \otimes \det^{-1}(F)$$

$$z \longmapsto z_b \otimes \bigwedge b_i^*$$

$F \simeq \mathbb{Z}[G]^d \quad b_1, \cdots, b_r, \underline{b_{r+1}, \cdots, b_d}$

claim. $(\psi_{r+1} \wedge \cdots \wedge \psi_d)(\underline{z_L}) = \pm\, \underline{\varepsilon_{K/k,S,T}}$

$\bigwedge^d P \longrightarrow \bigwedge^r P$ zeta element Rubin–Stark.

$\bigwedge^r \mathcal{O}^\times_{K,S,T} \subset \bigwedge^r P$

$\cup$

LHS

$$\boxed{\mathbb{Q}\bigwedge^r \mathcal{O}^\times_{K,S,T} \cap \bigwedge^r P = \bigcap_G^r \mathcal{O}^\times_{K,S,T}.}$$

$\mathrm{Fitt}_0(Cl_{\mathbb{Q}(\mu_{p^m})^+})$ also can describe $\mathrm{Fitt}_i(Cl_{\mathbb{Q}(\mu_{p^m})^+}) \quad i \geq 0$

$= I_\varepsilon.$