Euler systems: new constructions, results, and conjectures.

§1. Intro.

V, \( p \)-adic rep'n of \( \text{Gal} \).
unram. outside \( \mathcal{P} \subset \mathfrak{p} \subset \Sigma \)

ES for \( V = \text{classes} \)

\[ Z_m \in H^i(\mathbb{Q}(\mu_m), V^*(1)) \quad \forall m \]

satisfying a compatibility relation involving L-factors of \( V \).

**Note.** \( H^i(\mathbb{Q}(\mu_m), V^*(1)) \)
is a module over

\[ R_m = \mathbb{Q}_p \left( \mathbb{F}_m \right)^{x} \mathbb{Z} \] .

\[ \text{Observation (Gross):} \]

Why rank 2 BSD is difficult... If you want to construct a canonical element of a global Galois cohom. gp. that gp. had better to be of rank 1 (cover the relevant base ring.)
Suppose $V$ is odd (dim $V^c = 1 = \dim V^{c=-1}$)

(where $c = \text{complex conj.}$)

Then

$H^1(\mathbb{Q}(\mu_m), V^c(1))$ is "generically"

of rank $d/R_m$. (Euler-Poincaré)

+ Leopoldt issue

So if $d = 1$, we're in good shape

(e.g. Katz's ES).

What if $d > 1$?

Perrin-Riou: for $r \geq 1$, define

rank $r$ ES

= classes in $\Lambda^r H^1(\mathbb{Q}(\mu_m), V^c(1)) / R_m$

"Higher rank ES" Conjecture: $\equiv$ rank $d$ ES for $V$ odd of dim $2d$.

ES Hard to construct!
§2. Local conditions

Idea: Pick out a canonical rank 1 submodule
for classes to land in.

Do this with local conditions at $p$. Better to understand:

$$H^1(\mathfrak{a}, V^*(11)) \xrightarrow{loc_p} H^1(\mathfrak{a}_p, V^*(11))$$

$U_1$

$H^1_f(\mathfrak{a}_p, V^*(11))$ Block-Kato.

"Crystalline" classes

Fact. $\exists \pi \in H^1(\mathfrak{a}(\mathbb{C}m), V^*(11)) : \text{loc}_p \pi \text{ crystalline}$ at $\mathfrak{a}(\mathbb{C}m)$.

Has generic rank

$$d = \dim \left( \frac{D_{DR}(V)}{\text{Fil}^0} \right) = : r(V).$$

Conjecture. For $V$ coming from geometry (s.t. $r(V) \geq 0$)

$\exists$ ES for $V$ of rank $r(V)$,

for which all classes are crystalline at $p$.

related to $L^{(r(V))}(V, 0)$

(Note $L(V, s)$ vanishes to order $\geq r(V)$ at $0$.)
Rank 0 ES. = elts of $R_m$ related to $L$-values = a $p$-adic lift.

Can often choose $j$ s.t. $r(V_{cj})$ is 0 or 1.

(slide) $V = V_f(p) \otimes V_{c9}$

$r(V_{cj})$

\[ \begin{array}{cccccc}
2 & 4 & 6 & 7 & \cdots & j \\
\end{array} \]

\[ \begin{array}{c}
\text{construct ES} \\
\text{critical values}
\end{array} \]

§3. Refinements.

Want to use these ES's to bound Selmer groups

Defn. An $r$-refinement of $V$ is a subspace $V^c V$ of dim $d - r$ (where $V = 2d$-dim 'odd').

stable by $\text{Gal}$

(not nec. by $\text{Gal}$).
Say $V^+$ satisfies the Panchishkin condition if $V^+$ has all $HT \neq 2I$.

and all $HT \neq \frac{V}{V^+} \leq 0$.

(normalization: $HT \neq 0 \implies HT = 1$)

(This forces $r = r(V)$

and $V^+$ is unique if exists.)

Thm (Rubin + E. LLZ)

If $V$ has a $1$-refinement satisfying Panchishkin condition,

and

exists ES for $V$ that is crystalline at $p$

and $c_l \neq 0$

+ technical conditions,

then $Sel(\mathcal{O}, V) = 0$.

Refinements make sense in families (over rigid spaces/formal schemes)
Conjecture. If \( V \), a families over \( X \),

\[ V^+, \text{ 1-refinement over } X, \quad (r-) \]

and

\[ \exists \text{ Zariski-dense set in } X \]

where \( V^+ \) is Panchishkin,

then \( \exists \) family of rank 1 ES for \( V \)

with

local condition given by \( V^+ \).