Enhanced homotopy theory of the period integrals of hypersurfaces

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Glenn Stevens' 60th birthday, June 6, 2014

- I would like to thank the organizers and my thesis advisor Glenn Stevens and it is a real honor to be his student and talk at his 60th birthday conference.
- He suggested, as a thesis problem, a problem of finding p-adic theory of Weil (or oscillator) representation which would govern the p-adic θ-correspondence.
- But there was no genuine progress and I thought that one of the reasons is that I did not understand the physical nature or the role of the Weil representation in quantum field theory (no p-adic Haar measure $\sim$ no measure for the Feynmann path integral)
- By chance, I met a theoretical physicist Jae-Suk Park(the current collaborator) who had worked in string theory; he is an expert on QFT(quantum field theory). In particular, He made an **algebraic** formulation of QFT $\rightarrow$ may help to develop a p-adic theory of Schrödingier representation and Weil representation.
- The key language is the cochain complex enhanced with a binary product (correlations of Feynman path integrals) and induced infinite homotopy theory (notably, L_∞-homotopy theory and A_∞-homotopy theory).
- (Today's talk) We apply this algebraic formulation to period integrals of projective (or toric) hypersurfaces and complete intersections; we develop a (0+0)-dimensional quantum field theory QFT_X associated to such an algebraic variety X such that the period integral of X=the Feymann path integral in QFT_X.
- This leads us to a notion of a p-adic (0+0)-dimensional QFT which gives rise to the p-adic Dwork complex enhanced with a binary product (an ongoing project); this would shed some light (??) on p-adic properties of Weil representations? (maybe his 70th birthday)
- My recent mathematical journey - find higher homotopy structures appearing in number theory and algebraic geometry and try to find applications.

The goal of my talk is to reveal hidden structures on the Betti cohomology and period integrals of differential forms on smooth projective hypersurfaces over a field $\Bbbk$ in terms of BV(Batalin-Vilkovisky) algebras and L_∞-homotopy deformation theory.

I will concentrate on *BV algebras aspects* in this talk (I will **NOT** touch L_∞-homotopy related issues here).

A BV algebra over $\Bbbk$ (which arises from *a quantization scheme in physics*) is a cochain complex $(\mathscr{A} = \bigoplus_{m\in\mathbb{Z}} \mathscr{A}^m, K)$ over a field $\Bbbk$ equipped with a (super)commutative binary product $\cdot$ such that $(\mathscr{A}, K, \ell_2^K)$ is a differential graded Lie algebra and $(\mathscr{A}, \cdot, \ell_2^K)$ is a Possion graded algebra, where $\ell_2^K(a,b) := K(a\cdot b) - K(a)\cdot b - (-1)^{|a|} x K(b)$;

$$K\ell_2^K(a,b)+\ell_2^K(Ka,b)+(-1)^{|a|}\ell_2^K(a,Kb)=0, \quad \ell_2^K(a\cdot b,c)=(-1)^{|a|}a\cdot\ell_2^K(b,c)+(-1)^{|b|\cdot|c|}\cdot\ell_2^K(a,c)\cdot b.$$

Let n and d be positive integers. Let $\Bbbk = \mathbb{C}$.

- Let $X = X_G$ be a smooth projective hypersurface in the complex projective n-space $\mathbf{P}^n$ defined by a homogeneous polynomial $G(\underline{x}) = G(x_0, \cdots, x_n)$ of degree d in $\mathbb{C}[x_0, \cdots, x_n]$. Let $\Bbbk[X]$ be the homogeneous coordinate ring.
- Let $\mathbb{H} = H^{n-1}_{\text{prim}}(X, \mathbb{C})$ be the middle-dimensional primitive cohomology of X.
- The decreasing Hodge filtration $Fil^{\bullet}\mathbb{H}$ on $\mathbb{H}$.
- A cup product polarization on $\mathbb{H}$;

$$\mathbb{H} \otimes \mathbb{H} \to \mathbb{C}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta.$$

- For each $\gamma \in H_{n-1}(X, \mathbb{Z})$, we define a period integral of X;

$$C_{[\gamma]} : \mathbb{H} \to \mathbb{C}, \quad \omega \mapsto \int_\gamma \omega,$$

We will **enhance** all the above invariants *at the level of BV algebras.* As one application, we show how *such an enhancement (especially the binary product structure)* can be useful; we provide an explicit algorithm to compute the Gauss-Manin connections of families of smooth projective hypersurfaces of degree d.

- Let $S(y,\underline{x}) := y\cdot G(\underline{x})$, which we call *the Dwork potential*.

$$\mathscr{A} := \Bbbk[\underline{y}][\underline{\eta}] = \Bbbk[y_{-1},y_0,\cdots,y_n][\eta_{-1},\eta_0,\cdots,\eta_n], \quad y=y_{-1}, x_0=y_0,\cdots,x_n=y_n$$

$$K := \sum_{i=-1}^{n}\left(\frac{\partial S(y,\underline{x})}{\partial y_i}+\frac{\partial}{\partial y_i}\right)\frac{\partial}{\partial \eta_i}, \quad Q := \sum_{i=-1}^{n}\frac{\partial S(y,\underline{x})}{\partial y_i}\frac{\partial}{\partial \eta_i}, \quad \Delta := K-Q.$$

- We define three additive gradings on $\mathscr{A}=\Bbbk[\underline{y}][\underline{y}]$ with respect to the multiplication, called *ghost number* $gh\in\mathbb{Z}$, *charge* $ch\in\mathbb{Z}$ and *physical dimension* $pd\in\mathbb{Z}$, by the following rules;

$$\begin{aligned}
&gh(y_{-1})=0, \quad gh(x_j)=0, \quad gh(\eta_{-1})=-1, \; gh(\eta_j)=-1,\\
&ch(y_{-1})=-d, ch(x_j)=1, \quad ch(\eta_{-1})=d, \quad ch(\eta_j)=-1,\\
&pd(y_{-1})=1, \quad pd(x_j)=0, \quad pd(\eta_{-1})=0, \quad pd(\eta_j)=1,
\end{aligned}$$

  where $j=0,\cdots,n$. Then the ghost number is same as the cohomology degree.

- $$0\to\mathscr{A}^{-(n+2)}\xrightarrow{K}\mathscr{A}^{-(n+1)}\xrightarrow{K}\cdots\xrightarrow{K}\mathscr{A}^{-1}\xrightarrow{K}\mathscr{A}^{0}\to 0$$

- $$\mathscr{A} = \bigoplus_{gh,pd,ch}\mathscr{A}^{gh}_{(pd),ch} = \bigoplus_{-(n+2)\le j\le 0}\bigoplus_{w\ge 0}\bigoplus_{\lambda\ge 0}\mathscr{A}^{j}_{(w),\lambda}$$

$$Fil_m\mathscr{A} := \mathscr{A}_{(0)}\oplus\mathscr{A}_{(1)}\oplus\cdots\oplus\mathscr{A}_{(m)}, \quad Fil^m\mathscr{A} := Fil_{n-1-m}\mathscr{A}$$

Then $(Fil^{\bullet}\mathscr{A},K)$ becomes a filtered cochain complex.

A BV (Batalin-Vilkovisky) algebra associated to $\mathbb{H}$

Theorem (J. Park[2]) Assume that $d = n+1$ (for simplicity of presentation). Let $r := -dy\eta_{-1} + \sum_{i=0}^{n} x_i \eta_i$ and let θ be a formal element with $gh(\theta) = -2, ch(\theta) = 0, pd(\theta) = 1$. The quadruple $(\mathscr{A}_X := \mathscr{A}_0[\theta], \cdot, K_X = Q + r\frac{\partial}{\partial\theta} + \Delta, \ell_2^K)$ is a BV $\Bbbk$-algebra which satisfies

- The triple $(\mathscr{A}_X, \cdot, Q_X := Q + r\frac{\partial}{\partial\theta})$ is a commutative differential graded algebra (CDGA) and its cohomology is concentrated on degree (ghost number) 0 and $H_{Q_X}(\mathscr{A}_X) \simeq \Bbbk[X] \times \Bbbk$, where $\Bbbk[X]$ is a homogeneous coordinate ring of X.
- The cohomology $(\mathscr{A}_X, K_X)$ is also concentrated on degree 0 and is quasi-isomorphic to $(\mathbb{H}, 0) = (H^{n-1}_{\text{prim}}(X, \mathbb{C}), 0)$;
$$J : (\mathscr{A}_X, K_X) \longrightarrow (\mathbb{H}, 0).$$
- The quasi-isomorphism J sends the *pd* filtration $Fil^m \mathscr{A}_X$ to the Hodge filtration $Fil^m \mathbb{H}$. In fact, the filtered complex $(Fil^\bullet \mathscr{A}_X, K_X)$ gives rise to a spectral sequence whose E_1-term is $H_{Q_X}(\mathscr{A}_X)$ and E_2-term is $H_{K_X}(\mathscr{A}_X)$ which degenerates at E_2.

The binary product $\cdot$ in the cochain complex $(\mathscr{A}_X, K_X)$ has consequences on $\mathbb{H}$; the product $\cdot$ will induce a *formal Frobenius manifold structure* on $\mathbb{H}$, which gives the Gauss-Manin connection in a special case.

BV realization of the period integral and the polarization on $\mathbb{H}$

We define a $\mathbb{C}$-linear map $\oint : \mathscr{A}_X \to \mathbb{C}$ such that $\oint$ is a zero map on $\mathscr{A}_X^j$ if $j \neq 0$, otherwise;

$$\mathscr{A}_X \otimes \mathscr{A}_X \to \mathbb{C}, \quad u \otimes v \mapsto \oint u \cdot v, \quad \oint f := \frac{1}{(2\pi i)^{n+2}} \int_{X(\varepsilon)} \left(\oint_C \frac{f}{\frac{\partial S}{\partial x_0} \cdots \frac{\partial S}{\partial x_n}} y\, dy \right) dx_0 \wedge \cdots dx_n, \tag{1}$$

where C is a closed path on $\mathbb{C}$ with the standard orientation around $y = 0$ and

$$X(\varepsilon) = \left\{ \underline{x} \in \mathbb{C}^{n+1} \,\middle|\, \left| \frac{\partial G(\underline{x})}{\partial x_i} \right| = \varepsilon > 0, i = 0, 1, \cdots, n \right\}.$$

For each $\gamma \in H_{n-1}(X_G, \mathbb{Z})$, we define a $\mathbb{C}$-linear map $C_\gamma : \mathscr{A}_X \to \mathbb{C}$ such that C_γ is a zero map on $\mathscr{A}_X^j$ if $j \neq 0$, otherwise;

$$C_\gamma(u) := -2\pi i \int_\gamma Res \left(\int_0^\infty u \cdot e^{yG(\underline{x})} dy \Omega_n \right), \quad u \in \mathscr{A}_X^0 \tag{2}$$

where $\Omega_n = \sum_{i=0}^n (-1)^i x_i (dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n)$. Let $\mathscr{H}^n(X)$ be the n-th rational de Rham cohomology group of $\mathbf{P}^n$ regular outside X_G. (P. Griffiths) The residue isomorphism $Res : \mathscr{H}^n(X) \xrightarrow{\simeq} \mathbb{H}$.

Theorem (J. Park[2]) Under the isomorphism $J : H_{K_X}(\mathscr{A}_X) \xrightarrow{\simeq} \mathbb{H}$, (1) induces the cup product polarization on $\mathbb{H}$ and (2) induces the period integral $C_{[\gamma]}$.

Applications

Assume that $d=n+1$. Let $\{e_\alpha\}_{\alpha\in I}$ be a $\mathbb{C}$-basis of $\mathbb{H}=\bigoplus_{p+q=n-1}\mathbb{H}^{p,q}$ and divide $I=I_0\cup I_1\cup\cdots\cup I_{n-1}$ according to pd. Let $\{t^\alpha\}_{\alpha\in I}$ be its dual $\mathbb{C}$-basis. Let $y^k F_{[k]a}(\underline{x}), k\in I_a, a=0,\cdots,n-1$ be representatives of the cohomology classes $e_\alpha,\alpha\in I$.

$$\Gamma(\underline{t})=\sum_{a\in I_0} t_0^a F_{[0]a}(\underline{x})+y\cdot\sum_{a\in I_1} t_1^a F_{[1]a}(\underline{x})+\cdots+y^{n-1}\cdot\sum_{a\in I_{n-1}} t_{n-1}^a F_{[n-1]a}(\underline{x})\in\mathscr{A}_X^0\otimes\mathbb{C}[[\underline{t}]]. \quad (3)$$

There exists a unique 3-tensor $A_{\alpha\beta}^{\gamma}(\underline{t})\in\mathbb{C}[[\underline{t}]]$ (explicitly computable based on the Gröbner basis algorithm and depends only on L_∞-homotopy types of $\Gamma(\underline{t})$) such that

$$\partial_\alpha\Gamma(\underline{t})\cdot\partial_\beta\Gamma(\underline{t})=\sum_\gamma A_{\alpha\beta}{}^{\gamma}(\underline{t})\partial_\gamma\Gamma(\underline{t})+K_X(\Lambda_{\alpha\beta}(\underline{t}))+\ell_2^K(\Gamma(\underline{t}),\Lambda_{\alpha\beta}(\underline{t})),\quad \partial_\alpha=\frac{\partial}{\partial t^\alpha} \quad (4)$$

for some homotopy $\Lambda_{\alpha\beta}(\underline{t})\in\mathscr{A}_X^{-1}\otimes\mathbb{C}[[\underline{t}]]$. This PDE is a generalization of the Picard-Fuchs ODE. Then the following matrix of 1-forms with coefficients in power series in $\underline{t}_1$

$$A_\beta^\gamma(\underline{t}_1):=-\sum_{\alpha\in I_1} dt_1^\alpha\cdot A_{\alpha\beta}{}^{\gamma}(\underline{t})_{\Gamma(\underline{t})}\Big|_{t_j^a=0,a\in I_j,j\neq 1},\quad \beta,\gamma\in I,$$

becomes the connection matrix of formal Gauss-Manin connection along the geometric deformation $G_{\underline{t}_1}(\underline{x})=G(\underline{x})+\sum_{a\in I_1} t_1^a F_{[1]a}(\underline{x})$ given by the $\mathbb{H}^{n-2,1}$-component of the L_∞-homotopy type of $\underline{f}$. We have

$$\partial_\gamma\omega_\beta^\alpha(X_{G_{\underline{t}_1}})-\sum_{\rho\in I} A_{\gamma\beta}{}^{\rho}(\underline{t}_1)\cdot\omega_\rho^\alpha(X_{G_{\underline{t}_1}})=0,\quad \gamma\in I_1,\alpha\in J,\beta\in I,$$

where $\omega_\beta^\alpha(X_{G_{\underline{t}_1}})$ is the period matrix of a deformed hypersurface $X_{G_{\underline{t}_1}}$.

Happy Birthday! Glenn!