

6-4-2014 Ander Steele Glennfest.

10:45 AM.

The Shimura modular symbol and
critical slope Eisenstein series.

Background:

$$p=3, \ell=11.$$

$$\Gamma = \Gamma_0(33)$$

$M =$ a right Γ -module.

$$\left. \begin{array}{c} S_{\Gamma}(M) \\ \parallel \\ \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \\ \parallel \\ \text{Hom}_{\mathbb{Z}[\Gamma]}(\Delta_0, M). \end{array} \right\}$$

$$\text{Hom}_{\mathbb{Z}[\Gamma]}(\Delta_0, M).$$

$$\varphi$$

$$\varphi(xD)|_x = \varphi(D)$$

E.g. $M = \mathbb{Q}$ with triv. action.

$$S_{\Gamma}(\mathbb{Q}) = M_2(\Gamma) \oplus S_2(\Gamma).$$

↑
Eichler-Shimura.

Define $\varphi : \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{Q}$ by

$$\varphi(r) = \begin{cases} 1 & \text{if } r \in \Gamma_0(11) \cdot 0. \\ 0 & \text{if } r \in \Gamma_0(11) \cdot \infty. \end{cases}$$

$$\varphi \in S_T(\mathbb{Q}) \text{ by } \varphi\{r \rightarrow s\} = \varphi(r) - \varphi(s).$$

$$\cdot \varphi|_{T_g} = (1+g) \cdot \varphi \text{ for } g \neq 3, 11.$$

$$\cdot \varphi|_{u_{11}} = \varphi$$

$$\cdot \varphi|_{u_3} = 3 \cdot \varphi.$$

$\Rightarrow \varphi$ is the modular symbol
attached to Eisenstein series.

$$\underline{\text{slope} = 1}_4$$

$$\varphi \longleftrightarrow E_{2,11}^{\text{crit}}(z) = E_{2,11}(z) - E_{2,11}(3z)$$

$$\text{where } E_{2,11}(z) = \frac{l-1}{24} + \sum_{n \geq 0} \left(\sum_{\substack{d|n \\ l+d}} d \right) \cdot q^n$$

$D = \text{distributions on } \mathbb{Z}_p.$

$$D \twoheadrightarrow \mathbb{Q}_p$$

$$\mu \mapsto \mu(1) \text{ total measure.}$$

$$k-1=1.$$

$$\leadsto S_T(D) \xrightarrow[\varphi]{<1} S_T(\mathbb{Q}_p)^{<1}$$

(if cuspidal
period integral
 \rightarrow L-value.

Can we lift Eisenstein symbols to D ?

Conj. A (Pasol-Stevens) / Thm A (Bellaïche)

φ lifts uniquely to $\Phi_{Eis} \in S_T(D)$.

Conj. B (Pasol-Stevens) / Thm B (Bellaïche-Dargatzida) ^{Steele}

$$L_p(\Phi_{Eis}, \langle \rangle^s) = (1-s)(1-\langle 11 \rangle^{1-s}) \cdot \zeta_p(s) \cdot \zeta_p(2s).$$

Def. $L_p(\Phi, \sigma) = \int_{\mathbb{Z}_p^\times} \pm^{\sigma-1} d\mu(\pm)$

where $\mu := \Phi\{\infty, 0\}$.

and $\sigma \in \mathcal{X}(\mathbb{C}_p) = \text{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$.

How to construct?

Shintani modular symbol.

Let $V = \mathbb{Q}^2$.

* $\mathcal{S}(V) = \mathcal{S}(A_v^{(\infty)}) =$ linear combination of characteristic fns of affine lattices (\mathbb{Z} -valued) $\int GL_2 \mathbb{Z}$

e.g. $[\mathbb{Z}^2] = \bigotimes_{\mathfrak{f}} [\mathbb{Z}_{\mathfrak{f}}^2]$

$$* R = \mathbb{Q}((x, y)) / \substack{\text{quotient} \\ \mathbb{Q}} \hookrightarrow GL_2(\mathbb{Q})$$

$$\text{A cone } C = C(v, w), \quad v, w \in V \\ = \{ av + bw : a, b \in \mathbb{R}_{>0} \}$$

gives us a homomorphism

$$\mu_C : \mathcal{S}(V) \longrightarrow R \quad \text{by}$$

$$f \mapsto \sum_{v \in C \cap V} f(v) \cdot e^{v \cdot (x, y)} \\ \in \mathbb{Q}((x, y)) / \mathbb{Q}$$

$$\text{E.g.: } f(v) = [Z^2] \\ \text{cone } C\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\mu_C([Z^2]) = \sum_{n, m} e^{nx + my}$$

$$= \left(\frac{e^x}{1 - e^x} \right) \left(\frac{e^y}{1 - e^y} \right)$$

$$\in \frac{1}{x \cdot y} \cdot \mathbb{Q}[[x, y]]$$

Mellin transform ✓

Shintani zeta.

Get a $GL_2^+(\mathbb{Q})$ modular symbol

$$\mathfrak{I} : \Delta_0 \longrightarrow \text{Hom}(\mathcal{S}(V), R)$$

$$\{r, s\} \mapsto \mu_{C\{r, s\}}$$

orientation.



a Shimura modular symbol.

denote by \mathbb{I}^k the homog. deg k part of \mathbb{I} .

then $[\Phi^k] =$ Eisenstein measure of Stevens

= some period integral

of wt & Eisenstein series

on Borel-Serre \mathcal{H}^*

Fix $f' \in \mathcal{S}(A_v^{P,\infty})$, write T_f for $\text{stab}_{\text{SL}_2(\mathbb{Z})}(f')$

$$\simeq \text{Hom}(S(V), \mathbb{Q}[X, Y]) \xrightarrow{A'} \text{Hom}(S(V_p), \mathbb{Q}[X, Y])$$
$$D(\mathbb{Z}_p^2)$$

One can extend this to

$$\text{Hom}(S(V), R) \rightarrow \text{Hom}(S(V_p), R)$$

D , but invented $\rightarrow \tilde{D}(Z_p^2)$
certain differential operators.

$$D(\mathbb{Z}_p^2) \cap \mathbb{Q}[x, y] \supset S = \langle aX + bY \neq 0 \rangle$$

$$\text{by } (x \circ \mu)(f(x, y)) = \mu\left(\frac{\partial}{\partial x} f(x, y)\right)$$

similar for y .

$$\tilde{D} \subset \mathbb{Z}_p$$

$$\tilde{D} := S^{-1}D.$$

\uparrow

D

Get a modular symbol for $T_{f'}$

$$\Phi_{f'} : \Delta_0 \rightarrow \text{Hom}(\mathcal{S}(V), R) \xrightarrow{f'} \text{Hom}(\mathcal{S}(V_p), R)$$

\uparrow

$$\tilde{D}/\mathfrak{s}_0$$

Dirac at 0.

Thm. (Steele)

$$S_T(\tilde{D}(\mathbb{Z}_p \times \mathbb{Z}_p^\times))$$

$$\Phi_{f'} \in S_T(\tilde{D}(\mathbb{Z}_p^2)/\mathfrak{s}_0)$$

(after a suitable
modification at p .)

Cor. (Campbell, Kostadinov)

$\Phi_{f'}^k$ "vary analytically" in k .

Remark. Good choices of f
 \leadsto eigensymbols

Prop. \equiv $T_0(\mathbb{C}P^1)$ -equiv. hom.

$$S_T(\tilde{D}(Z_p \times Z_p^*)) \rightarrow \mathbb{S}(D_k(Z_p))$$

$$\Phi_{f'} \mapsto \Phi_{f'}^{(-2-k)} \leftarrow \theta\text{-operator used.}$$

$$f' = [Z_{11}^2] - 11 \cdot [Z_{11} \times 11Z_{11}] \otimes_{g \neq p, 11} [Z_g^2]$$

Thm. (Steele.)

$$L_p(\Phi_{f'}^{-2}, s) = L_p(\Phi_{Eis}^{crit}, s)$$

explicit computation
 \Downarrow
 confirm conjectures