6-6-2014. Benoit Stroh Glennfest 10:30AM.

Galois rep'ns and coherent cohomology

joint with Pilloni

based on Scholze's torsion paper.

modular form of wt $k \geq 2$.

$\rightsquigarrow$ Galois rep'n of HT wt $(0, k-1)$

regular (distinct HT wt)

modular form of wt 1.

$\rightsquigarrow$ HT wt $= (0,0)$ (Artin rep'ns)

$H^0_{\text{coherent}}(\text{modular curve}, \omega)$

Deligne-Serre: second line $(k=1)$ by p-adic interpolation of 1st line $(k \geq 2)$

(multiplication by E_{p-1}^{high})

Rem. H^1_{coherent} carries no new information.

Philosophy:

$H^j_{\text{coherent}}\left(\begin{smallmatrix}\text{Shimura}\\ \text{var.}\end{smallmatrix}\right) \circlearrowleft$ Hecke action.

the eigensystems should produce irregular Galois rep'ns.

Case of $U(r,s)$. relate to a quadratic CM ext'n of $\mathbb{Q}$

regular case in $H^0_{\text{coherent}} \longrightarrow$ Galois rep'ns of dim. $r+s$.

HT wt $= (\lambda_1 > \cdots > \lambda_{r+s})$

irregular case in H^0_{coherent}

$$\xrightarrow{?} \text{HT wt} = (\lambda_1 > \cdots > \underbrace{\lambda_r = \lambda_{r+1}}_{\text{very slightly irregular.}} > \cdots > \lambda_{r+s})$$

irregular case in H^j_{coherent}

$$\xrightarrow{?} \text{HT wt} = (\lambda_1 > \cdots > \lambda_r \,;\, \lambda_{r+1} > \cdots > \lambda_{r+s})$$

no relation.

Expect cohom. deg

$j = \#$ transposition $(i, i+1)$

that you need to order the list.

e.g. $U(n,n) \overset{?}{\rightsquigarrow} HT = (\lambda_1 = \lambda_1 > \lambda_2 = \lambda_2 > \cdots$
$> \lambda_n = \lambda_n)$

$$\left(\begin{array}{ll} \lambda_1 = \lambda_{s+1} = \lambda_{n+1} & \text{: notation} \\ \lambda_2 = \lambda_{n+2} & \end{array} \right)$$

II. Statements

- $X \longrightarrow \mathrm{Spec}(\mathbb{Z}_p)$ PEL Shimura variety.

$\uparrow$

A, abel. var. of genus g.

(of group unram. at p. no level at p. assoc. level N)

$$\begin{array}{c} A \\ \downarrow \uparrow e \\ X \end{array} \qquad \underset{\text{rk } g}{\underline{\Omega = e^* \Omega^1_{A/X}}} \qquad \underset{\text{rk } 1}{\underline{\omega = \det(\Omega)}}$$

- $\forall n$, $X_{\mathbb{Q}_p}(p^n) \longrightarrow X \times \mathrm{Spec}(\mathbb{Q}_p)$

full level for $A[p^n]$.

toroidal cptification. (Lan)

- X^{tor}, $X^{tor}_{\mathbb{Q}_p}(p^n)$

$\rightsquigarrow D$ boundary.

semi-abelian $\quad A \longrightarrow X. \longleftarrow \Omega, \omega.$

- $\mathcal{W} \longrightarrow X^{tor}$, autom. vector bundle.

(e.g. $\mathcal{W} = \mathrm{Sym}^2 \Omega$)

- $\mathbb{T} = \mathbb{Z}[\text{Hecke away from } Np]$.

classical quotient $\quad \mathbb{T}_{cl, \mathcal{W}, m} = \mathrm{Im}\left(\mathbb{T} \to \prod_{k>0} \mathrm{End}\, H^0(X^{tor}_{\mathbb{Q}_p}(p^m)), (\mathcal{W} \otimes \omega^k[-D])\right)$ $\quad (m \geq 0)$

classical regular ($k \gg 0$) cuspforms.

<u>Thm 1</u>. $\forall j > 0$, $\forall \underset{(\text{level})}{n} \geq 0$, $\forall m \geq \max(n, \frac{g}{p-1})$ (if $p = 2$, $\max(n, 2g)$),

$$\begin{tikzcd} \mathbb{T} \arrow[r, squiggly] \arrow[d] & H^j(X_{\mathbb{Q}_p}(p^m), \mathcal{W}(-D) \text{ or } \tilde{\mathcal{W}}) \\ \mathbb{T}_{cl} \arrow[ur, dashed] \end{tikzcd}$$

Remark.

- If Galois rep'n for regular H^0_{coherent}, same for H^i_{coherent}
- Related work of Boxer, Geraghty – Goldring.
- ramified PEL, Hodge type.

Assume X is compact.
(no toroidal cptification, no bdry)

Thm 2. (Scholze)

$\forall n \geq \frac{g}{p-1}$ ($2g$ if $p=2$),

$\exists\, k > 0$ and

① $\exists$ integral model.

Scholze's new model.

$$X(p^n)^{HT} \longrightarrow \mathrm{Spec}(\mathbb{Z}_p)$$

$\omega^{HT,k}$ ample. of $X_{\mathbb{Q}_p}(p^n) \longrightarrow \mathbb{Q}_p$.

ω^k line bundle $\leftarrow$ ample ($\Leftarrow$ cpt)

Hecke

② $\exists$

② $r = \binom{2g}{g}$

$\exists\, t_1, \dots, t_r \in H^0(X(p^n)^{HT} \times \mathbb{F}_p, \omega^{HT,k})$

s.t. • commute with Hecke.

• without base locus.

③ $\exists$ diagram of integral models

$$\begin{array}{ccccc} & & \mathcal{W} \searrow \ X(p^n)^{\text{modified.}} & & \\ & \mathcal{W} \ \ f \swarrow & & \searrow g & \\ \text{normalization of } X \text{ in } X_{\mathbb{Q}_p}(p^n) = & X(p^n) & & X(p^n)^{HT} & \end{array}$$

f, g blow up.

Thm 2 $\Rightarrow$ Thm 1.

need $\Pi \rightsquigarrow H^j(X_{\mathbb{Q}_p}(p^n), \mathcal{W})$

$$\Pi \searrow \Pi_{cl} \nearrow$$

assume
$$n \geq \frac{g}{p-1}.$$

enough $\Pi \rightsquigarrow H^j(X(p^n)^{HT}, g_* \mathcal{W})$

$$\Pi \searrow \Pi_{cl} \nearrow \text{Čech}$$

Scholze model.

Define $\mathcal{U}_i \underset{\text{open}}{\subset} X(p^n)^{HT} \times \mathbb{F}_p$

the locus where $t_i \neq 0$.

$$1 \leq i \leq r = \binom{2g}{g}.$$

it's

- affine $\Leftarrow \omega^{HT, tc}$ ample
- $\bigcup_{i=1}^{r} \mathcal{U}_i = X(p^n)^{HT} \times \mathbb{F}_p \Leftarrow$ no base locus.
- Hecke-stable.

$\rightsquigarrow$ It suffices to show

$$\cancel{\Pi \downarrow \Pi_{cl}} \qquad \cancel{H^0(\mathcal{U}_i, g_* \mathcal{W}}$$

$$\begin{tikzcd} \Pi \arrow[r] \arrow[d] & H^0(X(p^n)^{HT}, g_* \mathcal{W}/p) \\ \Pi_{cl} \arrow[ur, dashed] & \end{tikzcd}$$

enough.

$$\begin{tikzcd} \Pi \arrow[r] \arrow[d] & H^0(\mathcal{U}_i, g_* \mathcal{W}) \ni \varphi. \\ \Pi_{cl} \arrow[ur] & \end{tikzcd}$$

just. • φ can be extended to $X(p^n)^{HT} \times \mathbb{F}_p$.
by multiplication by E_i^{high} $\circlearrowleft$ Hecke.

• can be lifted to $X(p^n)^{HT}_{\mathbb{Q}_p}$

$\Leftarrow \omega^{HT, k}$ ample.

<u>Sketch of pf of thm 2.</u>

$g = 1$, modular curves.

$X(p^n) \longrightarrow \mathbb{Z}_p$ normalization of $\underset{\text{cpt.}}{\underset{\sim}{X}}$ in $X_{\mathbb{Q}_p}(p^n)$

<u>Have.</u>

$$E[p^n] \quad (X(p^n)) \xrightarrow{\alpha_E} \omega/p^n$$

$$\shortparallel$$

$$\mathbb{Z}/p^n e_1 \oplus \mathbb{Z}/p^n e_2.$$

(recall

$$\alpha : \underline{E[p^n]} \longrightarrow \omega_E/p^n$$

$$\shortparallel$$

$$\mathrm{Hom}(E[p^n], \mathbb{G}_m)$$

$$\varphi \longmapsto \varphi^*\left(\frac{dt}{t}\right)$$
)

Def. $s_1 = \alpha_G(e_1)$, $s_2 = \alpha_G(e_2)$

problem $\mathcal{O}s_1 + \mathcal{O}s_2 \subset \omega/p^n$

not isom. outside ordinary locus.

$$\begin{array}{ccc} & \uparrow & & \uparrow \\ \text{Def.} & \underline{\omega^{mod}} & \hookrightarrow & \omega \end{array}$$

just coherent. locally free rk 1 $/X(p^n)$.

Def. $X(p^n)^{mod} \to X(p^n)$

= normalization of blow up

where ω^{mod} is loc. free.

Get

$$\begin{array}{ccccc} & & X(p^n)^{mod} & & \\ & \swarrow & | & \searrow & \\ X(p^n) & & \omega^{mod} & & X(p^n)^{HT} \end{array}$$

line bundle.

Thm (Scholze) $\exists k$ s.t. $(\omega^{mod})^k$ generated by global sections.

Def. $X(p^n) \longrightarrow X(p^n)^{HT}$ = Stein factorization for $(\omega^{mod})^k$

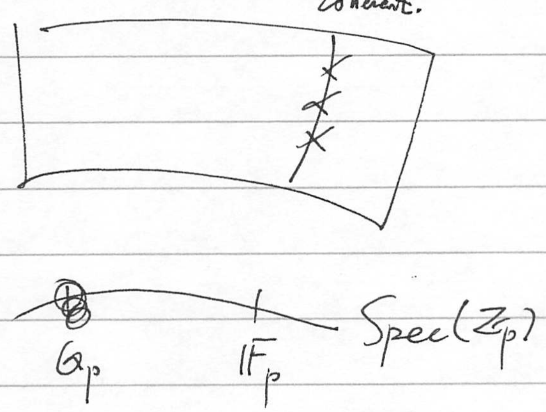
= best contraction where $(\omega^{mod})^k$ is ample.

$$\omega^{HT,k} \quad (= (\omega^{mod})^k)$$

$$= \mathrm{Proj}\left(\bigoplus_{r>0} H^0((\omega^{mod})^k)\right)$$

Picture $(g=1,\ X(p^n) \text{ smooth})$

ω^{mod} coherent. $\quad \times$ supersingular pts



$\omega^{mod} = \omega$ on ordinary locus.

identify ordinary pts with supersingular pts.

$\omega^{mod} = 0$.