6-6-2014 Jacques Tilouine Glennfest 9:30 AM

Big image of Galois rep'ns assoc. to Hida families of automorphic forms.
(jt with Hida in progress)

<u>History</u>

Serre 1972 : $E/\mathbb{Q}$ <u>not CM</u>.

$\rightsquigarrow \forall p$,

$\rho_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}_p)$

has open image and full for $\forall\forall p$.

provided that $\overline{\rho}$ is irred$_{G}$.

$\left[\begin{array}{l} (E,p) \\ \rightsquigarrow \exists\, l_{E,p} \neq 0 \subseteq \mathbb{Z}_p \\ \text{s.t. } \Gamma_{SL_2}(l_{E,p}) \subset \mathrm{Im}\, \rho_{E,p} \end{array} \right]$

Ribet 1974–1980 :

$f \in S_k(\Gamma)$, $k \geq 2$. (not θ-lift of Hecke char of imag quad field.)

not CM]

($f \not\equiv \theta(\lambda_K)$)

$\mathcal{O}_0$ = Hecke ring. completion

for any p,

$\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$; $\mathcal{O}' \subset \mathcal{O}$

"set of conjugates"

$\exists \alpha \in GL_2(\mathcal{O})$, $\exists\, \widetilde{l}_{f,\iota_p} \subset \mathcal{O}'$

s.t. $\alpha\, \Gamma_{SL_2}(\widetilde{l}_{f,\iota_p}) \cdot \alpha^{-1} \subset \mathrm{Im}(\rho_{f,\iota_p})$

depending on reduction

$\exists\, \alpha \in GL_2(\mathcal{O})$
$\exists\, \underbrace{l_{f,p}}_{} \subseteq \mathbb{Z}_p$ $\Big)$ $\alpha \Gamma_{SL_2}(l_{f,p}) \cdot \alpha^{-1}$

$\subset \mathrm{Im}(\rho_{f,\iota_p})$.

focus on this level
&
congruence ideals.

(provided that $\bar{\rho}$ is irred.)

Hecke eigensystem.

$$\begin{tikzcd}
h_k(\Gamma, \mathbb{Z}_p) \arrow[r, "\lambda_f"] \arrow[dr] & \mathcal{O} \arrow[r] & \mathcal{O}/C_f \\
 & h' \arrow[r] & h'/C'
\end{tikzcd}$$

$C_f \leftarrow$ congruence ideal of f. $\quad \mathcal{O}/C_f \supseteq h'/C'$

??

$$S_k = \mathbb{C}f \oplus^{\perp} S' \circlearrowleft h'$$

residually CM.

CM congruence ideal $\subseteq \mathcal{O}$.

non CM

$$\begin{tikzcd}
h_k(\Gamma, \mathbb{Z}_p) \arrow[r, "\lambda_f"] \arrow[dr] & \mathcal{O} \arrow[r] & \mathcal{O}/C_{f,CM} \\
 & h_{CM} \arrow[r] & h_{CM}/C_{CM}
\end{tikzcd}$$

$\mathcal{O}/C_{f,CM} \supseteq h_{CM}/C_{CM}$

$C'_{f,CM} = C_{f,CM} \cap \mathbb{Z}$
$\subseteq \mathbb{Z}$.

h_{CM}
$\circlearrowleft$
$\mathbb{C}f \oplus^{\perp} S^{CM} \oplus^{\perp} S''$

Exercise: $\mathfrak{l}_{f,p} \neq (1) \iff \mathfrak{c}'_{f,CM,p} \neq (1)$

Hida 2012: Big image of Galois rep'ns
and p-adic L-functions

(Replace cong. ideal by L_p)

$$\mathrm{len}_{\mathcal{O}}\left(\mathcal{O}/\mathfrak{c}_f\right) \sim \frac{L(\mathrm{Ad}^\circ f, 1)}{\Omega}.$$

Katz p-adic L-fn of CM fields.

$$\mathrm{len}_{\mathcal{O}}\left(\mathcal{O}/\mathfrak{c}_{f,CM}\right) \sim \gcd\left(\frac{L(\mathrm{Ad}^\circ f, 1)}{\Omega}, \frac{L(\frac{\lambda}{\lambda^c}, 1)}{\Omega_\lambda}\right)$$

($\to$ GSp_4 or higher ~~rk~~ case?
more than GL_2? ...)

Hida family: $p > 2$, $u = 1 + p$.

$$\lambda : h^\circ(N) \longrightarrow \mathbb{I} \subseteq K$$

($\mathbb{I} = \Lambda_1$ $\iff$ no congruence)

non CM. finite torsion free. fin. ext'n.

$$\Lambda_1 = \mathbb{Z}_p[[T]] \subseteq \mathcal{L}$$

non CM family
$\Updownarrow$
all f in the family are non CM
(but $\bar{\rho}_f$ ~~mod~~ can be CM.)

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{I})$$

$\bar{\rho}_\lambda$ irred.

λ, not CM.

- $\exists \delta \in \rho_\lambda(D_p)$ (img of decomp. gp.)

  δ has eigenvalue α, β in $\mathbb{Z}_p$

  s.t. $\alpha^2 \not\equiv \beta^2 \pmod p$

Galois level

- $\exists \underset{\neq 0}{l_\lambda} \subset \Lambda_1$, $\exists \alpha \in GL_2(\mathbb{I})$ s.t.

  $\alpha \cdot \Gamma_{SL_2}(l_\lambda) \cdot \alpha^{-1} \subset \mathrm{Im}\,\rho_\lambda$.

  $$L_{\lambda,CM}' \underset{\text{easy}}{\leq} \mathrm{ord}_P(l_\lambda) \underset{\text{tricky}}{\leq} 2 \cdot \mathrm{ord}_P(L_{\lambda,CM}')$$

  $GL(2)$ "via a formula"

for $\forall$ ht 1 prime $P \in \mathrm{Spec}(\Lambda_1)$

$G = GSp_4 \sim J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$

$p > 2$

$u = 1 + p$.

Hida-Hecke alg. $\rightarrow$ $h^\circ(N)$ = Hida-Hecke alg. for G finite torsion free over Λ_2.

where $\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]]$

$$\lambda : h^{\circ}(N) \longrightarrow \mathbb{I} \quad (\text{"Hida family"})$$

$\leftarrow$ domain

$$\begin{array}{ccc} h^{\circ}(N) & & \mathbb{I} \\ & \searrow \quad \nearrow & \\ & \Lambda_2 & \end{array}$$

finite
torsion-free.

arithmetic prime

$$\Lambda_2 \supseteq P_{k_1, k_2} = (1+T_1 - u^{k_1},\ 1+T_2 - u^{k_2}) \qquad \underline{k_1 \geq k_2 \geq 3.}$$

(in Siegel case. "Hecke eigensystem is better than q-exp'n?")

$$I_{\pi} \subset S_{k,\pi}.$$

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general $\nearrow$ $\exists\ \rho_\lambda : G_{\mathbb{Q}} \longrightarrow G(\mathbb{I}).$

interpolating ρ_{λ_s}.

not general $\nearrow$ $\underline{\bar{\rho}_\lambda \text{ irred}}$

Siegel cuspforms

$$S_{k_1, k_2} = S^{\text{Saito-Kurokawa}} \oplus^{\perp} S^{\text{Soudry}} \oplus^{\perp} S^{\text{Yosh}} \oplus^{\perp} S^{\mathrm{Sym}^3}$$
$$\oplus^{\perp} S^{\text{twYs}} \oplus^{\perp} S^{\text{general}}.$$

twisted Yoshida $\Leftrightarrow$ analogue of CM.

$\exists F/\mathbb{Q}$ real quad. $\leftarrow$ for $\theta(f)$

$N_G((GL_2 \times GL_2)^{\circ})$

(last 3 $\Rightarrow$ irred.)

shape

$$\begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \oplus \begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix} \oplus \begin{pmatrix} \times & & & \times \\ & \times & \times & \\ & \times & \times & \\ \times & & & \times \end{pmatrix} \oplus \mathrm{Sym}^3 \rho_f \oplus \theta(f) \oplus \text{general}$$

change

$(\mathbb{Q} \times \mathbb{Q})/\mathbb{Q}$

$\theta(f \times g)$ on $(GL_2 \times GL_2)^{\circ}$

$\mathfrak{g} = \mathrm{Lie}(G)$ Let λ' be a classical Hecke eigensystem.

$$\left\{\begin{array}{l} \rho_{\lambda'} \text{ is irred} \\ \mathrm{ad}^0(\rho_{\lambda'} \text{ is irred} \end{array}\right. \overset{\text{def.}}{\iff} \lambda' \text{ is Galois general}$$

Assume λ is Galois general, that is
$\exists p$ arithmetic s.t. ρ_{λ_p} is Galois general.

Prop. If λ' is Galois general, then
$\exists \alpha, \exists \ell' \subset \mathbb{Z}_p$ s.t. $\alpha \cdot \Gamma_{Sp_4}(\ell') \cdot \alpha^{-1} \subset \mathrm{Im}(\rho_{\lambda'})$.

Thm 1. $\exists\, \delta \in \rho_\lambda(D_p)$ with ~~$\mathbb{Z}_p$-algebra~~ $\mathbb{Z}_p$-eigenvalues
s.t. $\alpha(\delta) \not\equiv \alpha'(\delta) \bmod p$
$\forall \alpha \neq \alpha'$ roots of G.

$\Rightarrow \exists\, l_\lambda \subset \Lambda_2$

s.t. $\alpha \cdot \Gamma_{Sp_4}(l_\lambda)\, \alpha^{-1} \subset \mathrm{Im}\, \rho_\lambda$.

Thm 2: $\overline{\mathbb{Q}}\,\Gamma'_{\lambda,\,\text{tw}\,\gamma_0} \subset \underline{P} \iff \ell_\lambda \subset \underline{P}$.

where $\underline{P} \in \operatorname{Spec} \Lambda_2$, $\underline{P} \neq (p)$.

$\boxed{\text{Sketch of proof of Thm 1.}}$

2 steps $\lambda \xrightarrow{\rho} \lambda'$

1) $\forall_{\text{root}}\ \alpha$, $U_\alpha(\Lambda_2) \cap \operatorname{Im}(\rho_\lambda) \neq 1$.

2)
$$\begin{pmatrix} (1+T_1)\cdot(1+T_2) & & & * \\ & 1+T_1 & & \\ & & 1+T_2 & \\ & 0 & & 1 \end{pmatrix}$$

$\rightarrow$ lose multiplicity.