Local-global compatibility of regular algebraic cuspidal automorphic reps when $\lambda \neq p$.

1-dim'l ex. LCFT $\sim$ GCFT.

$F$, $\mathbb{F}$ field

$\nu$, place.

$\text{Art}_\nu : \mathbb{F}_\nu^\times \rightarrow \mathbb{W}_{\mathbb{F}_\nu}^{ab} = \{ \sigma \in \text{Gal}(\overline{\mathbb{F}_\nu}/\mathbb{F}_\nu)^{ab} \}

\int \int ...

\text{Art} : \mathbb{F}_\nu^\times \backslash \mathbb{A}_F^\times \rightarrow G_p^{ab}

Local-global for CFT.

$=$ commuting of this diagram.

Local Langlands for $GL_n$.

$p$, prime

$\not\equiv p$

$\mathbb{W}_{\mathbb{F}_\nu} = \{ \sigma \in \text{Gal}(\overline{\mathbb{F}_\nu}/\mathbb{F}_\nu) : \sigma|_{\mathbb{F}_{\nu, \text{Frob}}} \in \text{Frob} \}^{ab}$

$\sigma \mapsto \sigma|_{\mathbb{F}_{\nu, \text{Frob}}}$
A WD repn is a triple \((r, V, N)\) such that:

\[
\forall \sigma \in \mathbb{F}_r,
\quad r(\sigma) \cdot N \cdot r(\sigma)^{-1} = (\mathbb{F}_r \cdot \sigma) \cdot N.
\]

Grothendieck:

\[
\exists \text{ contd. Galois repn } \iff \exists \text{WD repns}. \\
\exists \text{ contd. Galois repn } \iff \exists \text{ WD repns over } \overline{\mathbb{Q}}_p.
\]

\((r, V, N)\) is Frobenius-s.s. \(\iff r\) is semisimple

\[
N = \log(T_p),
\]

\((r, V, N)\) is semisimple \(\iff r\) is semisimple and \(N = 0,\) for LLC\(\overline{\mathbb{Q}}_p\).

\[
\exists \text{ contd. repn of } G_{\mathbb{F}_r} \iff \exists \text{ WD repns over } \overline{\mathbb{Q}}_p \\
\quad \text{ over } \overline{\mathbb{Q}}_p \quad \iff \text{ WD where Frob eigenvalues are } \text{p-adic units.}\]
Thm (Harris-Taylor, Henniart)

\[ \text{irred. smooth repn of } GL_n(F_v) \leftrightarrow \text{Frob. ss WD repn of } W_{F_v} \text{ over } C \text{ of dim. } n. \]

Conjecture (Langlands, Fontaine-Mazur)

\[ \text{Cuspidal automorphic repn of } GL_n(\mathbb{A}_F), \text{ algebraic} \leftrightarrow \{ \text{irred. cont. Galois repsns } \rho : G_F \to GL_n(\overline{\mathbb{Q}}_p), \text{ unram. at primes } \}
\]

\[ \pi \mapsto R_p(\pi) \]

satisfying

\[ \pi = \otimes' \pi_v \mapsto R_p(\pi) : G_F \to GL_n(\overline{\mathbb{Q}}_p) \]

\[ \text{res to } v \quad \text{WD}(R_p(\pi) |_{G_{F_v}}) \]

\[ \pi_v \mapsto \text{rec} (\pi_v \otimes | \det |^{-\frac{1}{2}}) \]

\[ \text{WD}(R_p(\pi) |_{G_{F_v}}) \underset{Frob_{ss}}{=} \text{rec} (\pi_v \otimes | \det |^{-\frac{1}{2}}). \]
Thm. (Harris-Taylor Lan-Taylor-Thorne, Schulze)

\( F, \text{ CM or tot. real.} \)

\( \Pi, \text{ regular algebraic cuspidal automorphic repn of } \text{GL}_n(\mathbb{A}_F). \)

\[ \exists \, R_p(\Pi) : G_F \rightarrow \text{GL}_n(\mathbb{A}_p) \]

satisfies L-G compatibility

at unramified primes

\[ (\Pi, \text{ unram.}) \]

\[ (F, \text{ unram.}) \]

Thm (Varma)

LGC

For all primes \( \nu \neq p, \)

\[ \text{WD} \left( R_p(\Pi) \big|_{G_F} \right) \overset{\text{ss}}{=} \hat{\text{rec}} \left( \Pi_{\nu} \otimes \text{det}^{1-n} \right) \overset{\text{ss}}{=} \]

\[ H^{1-n}_c(S_h) \]

Note: previously, needed \( \Pi_{\nu} = \Pi^{c,\nu} \)

\( \text{(conjugate self-duality)} \)

\[ \uparrow \]

Strategy: p-adically interpolate the repns associated \( \Pi'_{\nu} = \Pi'^{c,\nu}. \)
(Similitude)

\[ \Pi \text{ on } GL_n. \quad (GL U(n, n) = \text{quasi-split unitary group}) \]

\[ \begin{pmatrix} 2n & \end{pmatrix} \quad (-I_n) \]

\[ P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \]

\[ L = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad (\sim \text{ Res } F \otimes GL_n. \big) \]

\[ M \in \mathbb{Z}_{\geq 0}. \]

\[ \Pi^*(m) = \text{Ind}_{P(A^\infty F)}^{GU(n, n)} (\Pi \otimes 1 \det m) \]

\[ X^{\text{min}} - \text{Shimura var. assoc. to } GU(n, n) \]

\[ \text{Thm (HLTT)} \]

\[ \Pi^*(m) \in H^0(X^{\text{min, ord}}, E_{\rho}) \]

\[ \text{where } \rho \text{ is "negative"} \]

\[ = p\text{-adic autom. forms of weight } \rho. \]

An analogue in $GL_2$: $f$, an overconv. eigenform of negative wt

\[ H^0 (X, (n)_{\text{ord}}, \omega^\infty (-\text{cusp})) \]

\[ \rho < 0 \]
Lemma. \( \forall r, \, \ell, \) positive integers
\[
\bigoplus_{j=1}^{\infty} H^0(X, \mathcal{E}_{\text{sub}}^{\min} \otimes \mathcal{E}_{\text{sub}}^{j \cdot p \ell - 1} (p-1))
\]

\[
\phi \downarrow
\]

\[
H^0(X, \mathcal{E}_{\text{sub}}^{\min, \, \text{ad}}) \otimes \mathbb{Z}/p\mathbb{Z}
\]

\[(GL_2) \uparrow \]

\[\exists q_i, \text{ classical eigenform of } \pi \leq \ell \cdot q_i \]
\[\text{s.t. } f \equiv q_i \pmod{p^r} \forall i.\]

\[\pi' \in H^0(X, \mathcal{E}_{p})\]

\[p' \text{ large enough } \iff \ell \geq 2.\]

\[\text{Fact. } \pi' = (\pi')^{c, \nu}.\]

Thm (Many people)
\[\exists R_p(\pi') : G_F \rightarrow GL_{2n}(\mathbb{Q}_p),\]

satisfying full \(L\)-\(G\) compatibility.
Unramified primes: analogue of $T_p$.

(Satake) $\exists T_v \text{ s.t. } T_v|_{\pi'} = \text{tr} \circ \text{rec}(B\mathbb{C}(\pi') \otimes \mathbb{A}^{1-2n})$
evaluated at Froby.

Ramified primes: Bernstein center (Comm)

(Chenevier): $\exists T_{\sigma/v} \text{ s.t. } T_{\sigma/v}|_{\pi'} = \text{tr} \circ \text{rec}(B\mathbb{C}(\pi') \otimes \mathbb{A}^{1-2n})$
well gp.

\[
\text{for all } \sigma \in \mathcal{W}_{F_v}.
\]

Pseudo characters:

$S = \text{ramified primes } + p.$

$\prod_{p \in S} = \text{unram. Hecke algebra}$
acting on $H^0(X^\min, E^\text{sub}_p)$

at $s \in \mathcal{E}_p$: Bernstein center

$C \rightarrow \text{End}(H^0(X^\min, E^\text{sub}_p))$
\[ \exists \text{ a pseudo repn. / char.} \]
\[ \mathbf{T} : \mathbb{G}_F \rightarrow \prod_{p \in \mathbb{P}} \mathcal{O}_p \]
\[ \text{Frob}_n \mapsto T_n \nu, \text{ unram.} \]
\[ \sigma_n \mapsto T_{\sigma_n} \nu, \text{ ramified} \]

\[ \sim \rightarrow \text{ get a pseudocharacter} \]
\[ \exists \mathbf{T} : \mathbb{G}_F \rightarrow \prod_{p \in \mathbb{P}} \mathcal{O}_p \]
\[ \text{Frob}_n \mapsto T_n \nu \rightarrow \text{tr}
\[ \sigma_n \mapsto T_{\sigma_n} \nu \rightarrow \text{tr}
\]

\[ \sim \rightarrow R_p(\sigma) : \mathbb{G}_F \rightarrow \mathbb{G}_{L_2 n}(\mathcal{O}_p) \]

\[ \text{satisfying semisimple LG compatibility.} \]
\[ BC_{\mathbb{G}_{L_2 n}(\mathcal{O}_p)}(\sigma) = (\sigma \oplus \sigma_c \circ \nu) \otimes \mathbb{C}^M \]

\[ WD \left( R_p(\sigma) \right)_{G_{F, \nu}}^{\mathbf{SS}} \cong \text{rec} \left( BC_{\mathbb{G}_{L_2 n}(\mathcal{O}_p)}(\sigma) \right)^{1 - \frac{n}{2}} \]

\[ \cong \text{rec} \left( \sigma \otimes \left( \mathbb{I} \oplus \mathbb{I} \right) \right)^{1 - \frac{n}{2}} \]

\[ \oplus \text{rec} \left( \sigma \otimes \left( \mathbb{I} \oplus \mathbb{I} \right) \right)^{1 - \frac{n}{2}} \]