6-6-2014 Ila Varma Glennfest 14:30pm

Local-global compatibility of regular algebraic cuspidal automorphic rep'ns when $\ell \neq p$.

<u>1-dim'l ex.</u> LCFT $\rightsquigarrow$ GCFT.

F, # field

v, place.

$$\begin{array}{ccc} \mathrm{art}_v : F_v^\times & \xrightarrow{\ \simeq\ } & W_{F_v}^{ab} = \{ \sigma \in \mathrm{Gal}(\overline{F_v}/F_v)^{ab} : \sigma|_{F_v^{ur}} \in \mathrm{Frob}^{\mathbb{Z}} \}. \\ \downarrow & \circlearrowleft & \downarrow \\ \mathrm{Art} : F^\times \backslash \mathbb{A}_F^\times & \longrightarrow & G_F^{ab} \end{array}$$

Local-global for CFT.

= commuting of this diagram.

<u>Local Langlands for GL_n.</u>

p, prime.

$v \nmid p$.

$$W_{F_v} = \{ \sigma \in \mathrm{Gal}(\overline{F_v}/F_v) : \sigma|_{F_v^{ur}} \in \mathrm{Frob}^{\mathbb{Z}} \}$$

$$\sigma \mapsto \sigma|_{F_v^{ur}} = \mathrm{Frob}^{\mathrm{val}(\sigma)}.$$

over $\overline{\mathbb{Q}}_p$ or $\mathbb{C}$

Def'n A WD rep'n is a triple (r, V, N)

$$r : W_{F_v} \to GL(V) \text{ open kernel}$$

$$N \in End(V)$$

such that

$\forall \sigma \in W_{F_v}$,

$$r(\sigma) . N . r(\sigma)^{-1} = (\# k)^{val(\sigma)} \cdot N.$$

Grothendieck:

$$\left\{ \begin{array}{c} \text{conti. Galois rep'n} \\ \text{of } W_{F_v}/\overline{\mathbb{Q}}_p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{WD rep'ns.} \\ \text{over } \overline{\mathbb{Q}}_p \end{array} \right\}.$$

(r, V, N) is Frob s-s. iff r is semisimple

"$N = \log(I_p^{\pm})$"

(r, V, N) is semisimple iff r is semisimple and $N = 0$.

for LLC/$\mathbb{C}$.

$$\left\{ \begin{array}{c} \text{conti rep'n of } G_{F_v} \\ \text{over } \overline{\mathbb{Q}}_p \end{array} \right\} \underset{WD}{\overset{\longleftrightarrow}{\to}} \left\{ \begin{array}{c} \text{WD rep'ns over } \mathbb{C} \\ \text{where Frob eigenvalues} \\ \text{are p-adic units} \end{array} \right\}$$

Thm (Harris-Taylor, Henniart)

$\exists$ bijection

$$\left\{\begin{array}{c}\text{irred. smooth} \\ \text{rep'n of } GL_n(F_v) \\ \text{over } \mathbb{C}\end{array}\right\} \underset{\text{rec}}{\overset{\longleftrightarrow}{\longrightarrow}} \left\{\begin{array}{c}\text{Frob. s-s } WD \text{ rep'ns} \\ \text{of } W_{F_v} \text{ over } \mathbb{C} \\ \text{of dim. } n.\end{array}\right\}$$

Conjecture (Langlands, Fontaine-Mazur)

$$\left\{\begin{array}{c}\text{Cuspidal automorphic rep'ns} \\ \text{of } GL_n(\mathbb{A}_F), \text{ algebraic}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irred. conti. Galois rep'ns} \\ \rho : G_F \to GL_n(\overline{\mathbb{Q}}_p)\end{array}\right\}$$

- unram. $\forall\forall$ primes
- de Rham at p

$$\pi \longmapsto R_p(\pi)$$

satisfying

$$\begin{array}{ccc}\pi = \otimes' \pi_v & \overset{R_p}{\longmapsto} & R_p(\pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_p) \\ \big\downarrow \text{res } v & & \big\downarrow WD(R_p(\pi)|_{G_{F_v}}) \\ \pi_v & \overset{\text{rec}}{\longmapsto} & \text{rec}(\pi_v \otimes |\det|^{\frac{1-n}{2}})\end{array}$$

$$\boxed{WD(R_p(\pi)|_{G_{F_v}})^{\text{Frob-ss}} \simeq \text{rec}(\pi_v \otimes |\det|^{\frac{1-n}{2}}).}$$

Thm (Harris-~~Taylor~~ Lan-Taylor-Thorne, Scholze)

F, CM or tot. real.

π, regular algebraic cuspidal automorphic rep'n of $GL_n(\mathbb{A}_F)$.

$\exists\ R_p(\pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$

satisfies L-G compatibility at unramified primes
$\begin{pmatrix} \pi, \text{unram.} \\ F, \text{unram.} \end{pmatrix}$

Thm (Varma)

For all primes $v \nmid p$,

LGC upto semisimplification.

$$WD\left(R_p(\pi)|_{G_{F_v}}\right)^{ss} \simeq rec\left(\pi_v \otimes |\det|^{\frac{1-n}{2}}\right)^{ss}.$$

Note: previously, needed $\pi = \pi^{c,v}$
(conjugate self-duality)
$\updownarrow$
$H^i_{et}(Sh)$

Strategy p-adically interpolate the rep'ns associated $\pi' = \pi'^{c,v}$.

(similitude)

π on GL_n. $\quad$ (G) $U(n,n)$ = quasi-split (similitude) unitary group

F^{2n} and $\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$L = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \left(\simeq \mathrm{Res}^F_{\mathbb{Q}} GL_n. \right)$$

$m \in \mathbb{Z}_{\geq 0}$.

$$\pi(m) = \mathrm{Ind}_{P(\mathbb{A}_F^{p,\infty})}^{GU(n,n)(\mathbb{A}_F^{\infty,p})} (\pi^\infty \otimes |\det|^m)$$

$X^{\min}$ - Shimura var. assoc. to $GU(n,n)$

<u>Thm</u> (HLTT)

$$\pi(m) \in H^0(X^{\min,\mathrm{ord}}, \mathcal{E}_\rho^{\mathrm{sub}})$$

where ρ is "negative"

= p-adic autom. forms of weight ρ.

[analogue in GL_2 : f, an overconv. eigenform of negative wt

$\cap$

$$H^0(X_1(N)^{\mathrm{ord}}, \omega^{\otimes k}(-\mathrm{cusp}))$$

$\boxed{k < 0}$

Lemma. $\forall r, k$, positive integers

$$\bigoplus_{j=r}^{\infty} H^0(X^{min}, \mathcal{E}^{sub}_{\rho + j\cdot p^{k-1}\cdot(p-1)})$$

$$\downarrow$$

$$H^0(X^{min,ord}, \mathcal{E}^{sub}_{\rho} \otimes \mathbb{Z}/p^k\mathbb{Z})$$

(GL_2) f

$\rightsquigarrow \exists g_i$, classical eigenform of wt k_i

s.t. $f \equiv g_i \pmod{p^r}$ $\forall i$.

$\pi' \in H^0(X^{min}, \mathcal{E}^{sub}_{\rho})$

ρ' large enough $\longleftrightarrow$ "$k \geq 2$".

Fact. $\pi' = (\pi')^{c,\vee}$.

Thm (many people)

$\exists R_p(\pi') : G_F \longrightarrow GL_{2n}(\overline{\mathbb{Q}}_p)$

satisfying full L-G compatibility.

Unramified primes: analogue of T_p.

(Satake) $\exists T_v$ s.t. $T_v|_{\pi'} = \mathrm{tr}(\mathrm{rec}(BC(\pi')_v \otimes \det^{\frac{1-2n}{2}}))$
evaluated at Frob_v.

ramified primes: Bernstein center. (comm)

(Chenevier): $\exists T_{\sigma,v}$ s.t. $T_{\sigma,v}|_{\pi'} = \mathrm{tr} \circ \mathrm{rec}(BC(\pi')_v \otimes \det^{\frac{1-2n}{2}})_\lambda(\sigma)$.
($\sigma \in$ Weil gp.)

for all $\sigma \in W_{F_v}$.

Pseudocharacters:

S = ramified primes + p.

$\mathbb{T}^S_\rho$ = unram. Hecke algebra
acting on $H^0(X^{min}, \mathcal{E}_\rho^{sub})$

&

at $S \setminus \{p\}$: Bernstein center
$\hookrightarrow \mathrm{End}(H^0(X^{min}, \mathcal{E}_\rho^{sub}))$.

] $\mathbb{T}^p_\rho$ Hecke alg. away from p.

∃ a pseudo rep'n./char

$$T : G_F \longrightarrow \mathbb{T}^{p}_{\rho + j p^{k-1}(p-1)}$$

$$\mathrm{Frob}_v \mapsto T_v \qquad v, \text{ unram.}$$

$$\sigma_v \mapsto T_{\sigma,v} \qquad v, \text{ ramified}$$

⇝ get a pseudocharacter

$$\exists\, T : G_F \longrightarrow \mathbb{T}^{p,\mathrm{ord}}_{\rho} \xrightarrow{\pi(m)} \overline{\mathbb{Q}}_p$$

$$\mathrm{Frob}_v \mapsto T_v \longmapsto \mathrm{tr} \circ \mathrm{rec}(\cdots)$$

$$\sigma_v \mapsto T_{\sigma,v} \longmapsto \mathrm{tr} \circ \mathrm{rec}(\cdots)$$

⇝ $R_p(\pi(m)) : G_F \longrightarrow GL_{2n}(\overline{\mathbb{Q}}_p)$

satisfying semisimple LG compatibility.

$$BC^{GL_{2n}}(\pi(m)) = (\pi \oplus \pi^{c,\vee}) \otimes \varepsilon_{\rho}^{m}.$$

$$WD(R_p(\pi(m))|_{G_{F_v}})^{ss} \simeq \mathrm{rec}(BC(\pi(m)) \otimes |\det|^{\frac{1-2n}{2}})^{ss}$$

$$\simeq \mathrm{rec}(\pi_v \otimes |\det|^{\frac{1-n-m}{2}})$$

$$\oplus\, \overline{\mathrm{rec}(\pi_v \otimes |\det|^{\frac{1-n+m}{2}})}^{c,\vee}.$$