6-5-2014 Sarah Zerbes Glennfest 9:30AM

Euler systems and the Birch and Swinnerton-Dyer conjecture.

(jt. with Kings, Lei, Loeffler)

Building on BDR ← Rotger.

0. Motivation

Happy birthday, Glenn!

1. The BSD conjecture

$E_{/\mathbb{Q}}$, elliptic curve

$$\Rightarrow \quad E(\mathbb{Q}) \simeq \underbrace{\Delta}_{\text{finite}} \times \mathbb{Z}^{r_E} \quad \leftarrow \text{rank of } E.$$

$L(E,s)$ Hasse-Weil L-function. (as the Euler product of local info.)

converges for $Re(s) > 3/2$.

Thm (BCDT, 2001)
$L(E,s)$ has analytic conti. to $\mathbb{C}$.

BSD conj.: $\cdot \operatorname{ord}_{s=1} L(E,s) = r_E$.

$\cdot$ leading term can be expressed in terms of arithmetic invariant of E (including $\#Ш(E/\mathbb{Q})$)

Generalization:

ρ = Artin rep'n of $G_{\mathbb{Q}}$ factoring through F.

$\rightsquigarrow L(E, \rho, s)$

BSD (ρ) conjecture.

$$\operatorname*{ord}_{s=1} L(E, \rho, s) = \operatorname{rank} E(F)[\rho].$$

Thm. (Kolyvagin, Kato)

If $L(E, 1) \neq 0$, then $E(\mathbb{Q})$ is finite and $Ш(E/\mathbb{Q})[p^\infty]$ is finite for almost all p.

Key ideas (following Kato)

① make problem p-adic

- attach to E a p-adic rep'n of $G_{\mathbb{Q}}$.

$$Ta_p \, E = \varprojlim_n E(\overline{\mathbb{Q}})[p^n] \quad (\simeq \mathbb{Z}_p^2).$$

$$V_p E = Ta_p E \otimes \mathbb{Q}_p.$$

-]

- Consider $G_{\mathbb{Q}}$-cohomology

$$\rightsquigarrow H^1(\mathbb{Q}, V_pE) := H^1(G_{\mathbb{Q}}, V_pE).$$
$$\cup\!\!\!\mid$$
$$\mathrm{Sel}(\mathbb{Q}, V_pE)$$

Facts: * $E(\mathbb{Q}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p \hookrightarrow \mathrm{Sel}(\mathbb{Q}, V_pE)$.

* quotient is related to $Ш(E/\mathbb{Q})[p^\infty]$.

$\Rightarrow$ Sufficient to show:

$$L(E,1) \neq 0 \Rightarrow \mathrm{Sel}(\mathbb{Q}, V_pE) = 0.$$

② Construct an Euler system (ES) for V_pE.

- collection of classes $(z_m)_{m \geq 1}$

$$z_m \in H^1(\mathbb{Q}(\mu_m), V_pE).$$

satisfying certain compatibilities under ~~coe~~ cores. maps.

ES is related to $L(E,1)$

$\exists$ linear functional

$$\exp^*: \underbrace{H^1(\mathbb{Q}_p, V_pE)}_{\text{local}} \longrightarrow \mathbb{Q}_p.$$

$$\text{s.t. } \exp^*(z_1) = \frac{L(E,1)}{\Omega}.$$

$$(L(E,1) \neq 0 \Rightarrow z_1 \neq 0)$$

③ Use duality theorems in Galois cohomology to show:

$$\exp^*(z_1) \neq 0 \Rightarrow Sel(\mathbb{Q}, V_p E) = 0.$$

II. Euler systems.

Def. (Rubin)

Let K = number field

V = p-adic rep'n of G_K.

unramified outside finite set of primes.

$\Sigma \ni p$.

An ES for (K, V) is a collection of classes (z_m), m integral ideal of K.

$$z_m \in H^1(\underbrace{K(m)}_{\text{ray class field mod } m}, V^*(1))$$

s.t. · z_m lands in fixed lattice $T \subset V^*(1)$ indep of m (bddness)

ES norm relation

· $\mathrm{cor}^{K(lm)}_{K(m)} z_{ml} = \begin{cases} z_m & \text{if } l \mid m \text{ or } l \in \Sigma. \\ P_l(\sigma_l^{-1}) \cdot z_m & \text{otherwise.} \end{cases}$

where σ_ℓ = arithmetic Frob at ℓ.

$P_\ell(x) = \det(1 - \sigma_\ell^{-1} \cdot X | \underset{\text{not } V^*(1) \text{ !!}}{\underline{\underline{V}}})$

<u>Remarks.</u>

- need to consider $V^*(1)$ because of duality theorems

- If $V = V_p E$, then $V^*(1) \simeq V$.

- If $z_1 \neq 0$, get bounds of $\mathrm{Sel}(K, V)$ or related Selmer gps.

even $\mathrm{Sel} \neq 0$ case!

<u>Conj.</u> A nonzero ES should exist whenever V comes from geometry.

<u>Known cases.</u>

twist by Dirichlet char.

(1) $K = \mathbb{Q}$, $V = \mathbb{Q}$, cyclotomic units.

(2) K = img. quad, $V = \mathbb{Q}$, elliptic units

(3) $K = \mathbb{Q}$, $V = V_p E$ or $V = V_p(f)$, Kato's ES.

ring class field not ray class field

[(4) K img. quad or CM, $V = V_p E$, Heegner pts / Heegner cycles.]
(Kolyvagin) (Nekovar)

<u>Thm</u> (LLZ, KLZ)

Let f, g be modular forms of wts $k+2, k'+2 \geq 2$
of level N $(p \nmid N)$

Let $0 \leq j \leq \min(k, k')$.
Define
$$V = V_p f \otimes V_p g\,(1+j)$$
Then there exists classes

Beilinson -Flach
$$BF_m^{(f,g,j)} \in H^1(\mathbb{Q}(\mu_m), V^*(1))$$
satisfying ES-type norm relations.

<u>Relation to L-values</u>

$$BF_1^{(f,g,j)} \in \ker(\exp^*),$$
but $\exists \log : \ker(\exp^*) \longrightarrow \mathbb{Q}_p$.
$$\log(BF_1^{(f,g,j)}) = (*) \cdot \underbrace{L_p(f,g,1+j)}_{L_p(V,0)}$$

$L_p(V,0) \leftarrow$ ~~Hida family of~~ f, g. ??
Hida-Rankin L-ftn.

idea. regulator $\mathbb{C}$ and p-adic $\leftarrow$ not in the critical range

Fact · elliptic curve $\longleftrightarrow$ wt 2 mod. forms

$$V_p E = V_p(f)(1).$$

· 2-dim'l Artin rep'n

$\longleftrightarrow$ wt 1 mod form.

$\Rightarrow$ don't get ES for $V_p(E)(\underset{\text{wt } 1 < 2.}{\underline{\rho}})$

Idea. Use p-adic deformation.

III. A 3 variable ES.

$\mathcal{f}, \mathcal{g}$ Hida families.

$\Lambda_{\mathcal{f}}, \Lambda_{\mathcal{g}}$: localization of Hecke algebra.

$M_{\mathcal{f}}, M_{\mathcal{g}}$: Λ-adic rep'ns attached to $\mathcal{f}, \mathcal{g}$.

$\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Thm (KLZ)

$\forall m \geq 1,\ p \nmid m,\ \exists\ BF_m^{(\mathcal{f},\mathcal{g})} \in H^1(\mathbb{Q}(\mu_m), \overbrace{M_{\mathcal{f}}^* \hat{\otimes} M_{\mathcal{g}}^* \hat{\otimes} \Lambda(\Gamma)}^{\text{3 variables.}})$

s.t. - they satisfy ES type norm relations.

- If f, g are in $\mathcal{f}, \mathcal{g}$ of wt $k+2, k'+2$ (≥ 2), and $0 \leq j \leq \min(k, k')$,

then specialization of $BF_m^{(\mathcal{F},\mathcal{G})}$
at ~~and~~ (f, g, j)
recovers $BF_m^{(f,g,j)}$

- these pts (f, g, j) are dense in Hida family, so get a relation to p-adic L-values everywhere even ~~at~~ critical pts.

$E/\mathbb{Q} \longleftrightarrow f.$ } ord at p.
ρ, 2-dim'l Artin rep'n $\longleftrightarrow g.$ }

Let $\mathcal{F}, \mathcal{G}$ be Hida families through f and g.

$\rightsquigarrow \left(BF_m^{(\mathcal{F},\mathcal{G})}\right)_{m\geq 1}$

$\rightsquigarrow$ specialize at $(f, g, 0)$
$\rightsquigarrow$ get ES for $V_pE(\rho)$ related to $L(E, \rho, 1)$

Theorem (KLZ)

Let $p \geq 5$, assume E does not have CM, ordinary at p.

let ρ = dihedral Artin character
(induced from img. quad field K)
factoring through F.

Assume that p splits in K + technical hypothesis
(~~satisfyin~~ satisfied for infinitely many p).

If I

If $L(E, \rho, 1) \neq 0$,
then both $E(F)[\rho]$ and the p-primary part of $\text{Ш}(E/\mathbb{Q})[\rho]$ are finite.

Remarks.

- finiteness of $E(F)[\rho]$ was first proven by Bertolini–Darmon–Rotger. (ρ not nec. dihedral)
- If ρ is anticyclotomic, this was proven by Longo–Vigni.

Mazur — Construction of classes?

$$Y_1(N) \qquad Y_1(Nm^2) \longrightarrow Y_1(N)^2 \times_{\mathbb{Q}} \mathbb{Q}(\mu_m)$$

$$\mathcal{O}^{\times}_{Y_1(N)} \qquad z \longmapsto \left(z, z+\frac{1}{m}\right) \in \mathfrak{h}$$

$$\text{Siegel units} \qquad CH^2(Y_1(N)^2 \times \mathbb{Q}(\mu_m), 1)$$

twist by 1.

Mazur — $f = g$? → decomposable rep'n

$\downarrow$

Sym^2 maps to 0....

→ consider the derivative?