Stanisław Ulam

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Stanisław Marcin Ulam (1909–1984) was a brilliant, pioneering and eclectic mathematician, very much the fox rather than the hedgehog. In his early years in Poland he provided a number of seminal results in the budding fields of set theory, topology, measure theory and functional analysis. Established at Los Alamos in 1944, he soon made a crucial contribution to the development of the hydrogen bomb and, with the development of computers, inspired and named the Monte Carlo method of statistical sampling. In later years he made important contributions to the study of fluid dynamics and non-linear and ergodic systems.[a]

The Ulam–Gödel correspondence is sparse and sporadic, with Ulam raising mathematical issues or making inquiries and Gödel writing always in reply. The correspondence does cast some light, albeit from the periphery, on several issues in the historical development of set theory in its formative period.

The first several letters have as a common thread possibilities for having measures on collections of sets. For present purposes, a ***measure*** on a collection $\mathcal{F}$ is a function $m : \mathcal{F} \longrightarrow [0,1]$, the unit interval of reals, such that m is not identically zero, $m(\{x\}) = 0$ for any singletons $\{x\} \in \mathcal{F}$, and m is countably additive, i.e. for pairwise disjoint $\{X_n \mid n \in \omega\} \subseteq \mathcal{F}$, $m(\bigcup_n X_n) = \sum_n m(X_n)$. In the fundamental paper *Ulam 1930*, drawn from his dissertation, Ulam had investigated the possibility of having a measure on the full power set $\mathcal{P}(X)$ of some set X, and the concept of a ***measurable cardinal***, the most important of all concepts in the theory of large cardinals in set theory, emanates directly from this paper. A cardinal κ is ***measurable iff*** there is a measure on $\mathcal{P}(\kappa)$ with range $\{0,1\}$, i.e. the measure is two-valued.

In letter 1, the initial letter of 16 January 1939, and with more detail in letter 2, the succeeding letter of three years later, Ulam refers to their "joint remark" that it is not possible to find a measure on the collection of projective sets of reals. The reconstruction is straightforward: *Ulam 1930* established that there is no measure on the power set $\mathcal{P}(\aleph_1)$ by using a doubly indexed system of sets now known as an *Ulam matrix*. Gödel established that the restriction of the canonical well-ordering of L to its reals is a projective, in fact Σ^1_2, well-ordering. He derived from this that if $V = L$, then there is a projective, in fact Δ^1_2, set of reals that is not Lebesgue measurable. Using Gödel's projective well-ordering one can in fact generate an Ulam matrix consisting of projective sets

[a] *Ulam 1974* is a collection of selected papers; *Ulam 1976* is an autobiography; and *Cooper 1989* is a festschrift.

and thereby draw the following conclusion generalizing Gödel's non-Lebesgue-measurability result: If $V = L$, then there is no measure at all on the collection of projective sets.[b]

Ulam in letter 1 writes that he thinks that their joint remark makes it "imperative to change the problem of measure!" and asks whether the remark is worth publishing. In letter 2 Ulam inquires "whether you ever published your results on the non-measurable projective sets or our joint remark." In letter 3, his undated response, Gödel somewhat cryptically says no. In fact, only in a footnote of his later, 1951 edition of *Gödel 1940*, his monograph on L, did he sketch his result that if $V = L$, then there is a Σ^1_2 well-ordering of the reals.[c] In any case, Ulam's inquiries tend to undermine Kreisel's eye-catching remark in his memoir on Gödel that "... according to Gödel's notes, not he, but S. Ulam, steeped in the Polish tradition of descriptive set theory, noticed that the definition of the well-ordering ... of subsets of ω was so simple that it supplied a non-measurable PCA [i.e. Σ^1_2] set of real numbers ..."[d]

Ulam's letter of 14 August 1942, letter 2, actually begins with an inquiry about an idea for establishing that the existence of a non-Lebesgue-measurable projective set of reals already follows from the continuum hypothesis (CH), or possibly from the weaker hypothesis that the power of the continuum is less than the least weakly inaccessible cardinal.[e] Ulam includes a sketch (Gödel *Nachlaß* document 012880) of his idea, which turns on another use of Ulam matrices in a projective setting. If successful, the idea would replace Gödel's $V = L$ assumption for getting a non-Lebesgue-measurable projective set simply by a cardinal hypothesis. In his reply, letter 3, Gödel points out a problem already at the second step of the sketch. He also points out that because Ulam's proposed construction is effective, if successful, it would establish the existence of a non-Lebesgue-measurable projective set without appealing to the axiom of choice, "which is unlikely." On the margin he writes "I have a feeling that the measurability of all proj. sets is consist. log. with the ax. of choice."

Ulam later published the idea sketched in 012880 in his collection of problems. *Ulam 1960*, p. 17ff. With the advent of forcing, Gödel's "feeling" turned out to be vindicated: *Solovay 1970* established that if ZFC + "there is an inaccessible cardinal" is consistent, then so is ZFC + CH + "all projective sets are Lebesgue measurable". Moreover, CH here can be replaced by $2^{\aleph_0} = \aleph_\alpha$ for a wide range of α's.[f]

Ulam's letter of 14 August 1942 is significant in other respects. In it is also a remarkable anticipation of effective descriptive set theory. Alluding to some remarks of Nikolai Luzin about "really" constructive Borel sets, Ulam formulates topological concepts in terms of the recently developed mathematical concept of recursive set of integers. This is a very early example of the formalization of effectiveness in terms of recursiveness. Ulam describes the open, closed and G_δ sets of reals with recursive codes and notes that one can proceed into the finite Borel classes and "even some transfinite ordinal classes." He then inquires whether, as with the classical Borel hierarchy, new sets appear at each level of the effective version and also of the effective versions, analogously defined, of the projective hierarchy. In effect, Ulam is asking whether what are now known as the "lightface" arithmetical and analytical hierarchies are proper.[g]

Gödel in letter 3 astutely points out, in effect, that the universal sets of classical descriptive set theory used to establish the properness of hierarchies are "lightface"—the implicit reply to Ulam's question is therefore in the affirmative. This exhibits a comparative expertise in descriptive set theory on Gödel's part, a circumstance that further militates against Kreisel's remark quoted above.

The actual mathematical development would proceed in reverse order: The arithmetical hierarchy was developed in *Kleene 1943* and the analytical hierarchy in *Kleene 1955*, *1955a* and *1955b* in terms of formula complexity and shown to be proper. The connection through effective topology with the hierarchies of classical descriptive set theory was only made afterwards in *Addison 1958*.

Finally, Ulam in letter 2 discusses incorporating projection as an operation into Boolean algebras involving Cartesian products, as a way of treating quantification on an algebraic basis. This would be developed in his study of *projective algebras* with his friend and collaborator C. J. Everett in *Everett and Ulam 1945*.[h] Gödel in his undated reply sagaciously makes a connection to the work of Ernst Schröder and moreover raises "the problem of decision". This is an early speculation for algebraic theories; the theory of Boolean algebras itself was shown to be

[b]For the projective sets of reals and their properties in L see Solovay's note in these *Works*, vol. II, p. 13ff., or *Kanamori 1997*, §§12 and 13. See also the latter, §2, p. 24, for the Ulam matrices argument.

[c]See these *Works*, vol. II, p. 33.

[d]*Kreisel 1980*, p. 197.

[e]A *weakly inaccessible* cardinal is an uncountable regular limit cardinal.

[f]See *Kanamori 1997*, §11 for an exposition of Solovay's result. The hypothesis that there is an inaccessible cardinal is nowadays considered a very mild assumption. In any case, *Shelah 1984* established that it is necessary in Solovay's result, by showing that if ZFC + "every Δ^1_3 set of reals is Lebesgue measurable" is consistent, then so is ZFC + "there is an inaccessible cardinal".

[g]See *Kanamori 1997*, §12 for effective descriptive set theory.

[h]Projective algebras are precursors of the better known *cylindric algebras* of *Henkin, Monk and Tarski 1971* and *1985*.

decidable by *Tarski 1949a*.[i] Overall, Gödel's reply is remarkable for addressing each of Ulam's concerns by getting to the heart of the matter with an economy of words.

Ulam in letter 4, dated 6 December 1947, revisits his analysis of measure in *Ulam 1930* in connection with the power of the continuum. He first praises Gödel's recent article on the continuum problem, *Gödel 1947*; notes that he (Ulam) too believes in the falsity of CH in the sense that new axioms will be added that, e.g., will entail that $2^{\aleph_0}$ is the least weakly inaccessible cardinal; and recalls discussing possibilities of such axioms with Gödel in 1938 and later.

Ulam then proceeds to sketch an approach via measure. He first notes that he had shown in *Ulam 1930* that for no cardinal λ less than the least weakly inaccessible cardinal is there a measure on the power set $\mathcal{P}(\lambda)$. He then writes "I believe that it is possible" for the least weakly inaccessible cardinal κ that there *is* a measure on $\mathcal{P}(\kappa)$. For this he envisions a transfinite construction procedure similar to Gödel's for his L and suggests the likelihood of postulating an axiom asserting the existence of more and more sets of reals "definable by transfinite induction," entailing that $2^{\aleph_0}$ is the least weakly inaccessible cardinal.[j] Ulam concludes his letter with speculations about certain collections of sets of reals and a possible axiom about such collections never being the full power set $\mathcal{P}(\mathbb{R})$.

Whether there could be a measure on $\mathcal{P}(\kappa)$ for κ the least weakly inaccessible cardinal would remain an open problem for several decades after *Ulam 1930* until Solovay in 1966 (see *Solovay 1971*) established that this is false in a strong sense.[k] Ulam's letter of 25 October 1957, letter 5,[l] is part of an interchange, most of it unpublished here, regarding his memoir on his recently deceased friend, John von Neumann. Ulam writes to Gödel somewhat flippantly: "I have always regarded as the principal value of von Neumann's articles on axiomatization, the fact that you used his system or a similar one for some of your work!" Gödel

[i]Tarski does state at the end of the abstract that its results were obtained in 1940.

[j]Note that this measure approach to the size of the continuum complements and yet is consistent with his speculations at the beginning of letter 2 about the existence of non-Lebesgue-measurable sets of reals under the assumption that $2^{\aleph_0}$ is less than the least weakly inaccessible cardinal.

[k]The least cardinal κ for which there is a measure on $\mathcal{P}(\kappa)$ is the least "real-valued measurable cardinal", and Solovay established just from a "saturated ideal" property of such a cardinal κ that it must be the κth weakly inaccessible cardinal and a fixed point in the sequence of cardinals in various strong senses. See *Kanamori 1997*, §§2 and 16, particularly 16.8.

[l]Archives of the American Philosophical Society.

in his reply of 8 November 1957, letter 6,[m] warmly demurs and proceeds to make several remarks of considerable import on von Neumann's work on the axiomatization of set theory. First, Gödel considers that von Neumann's "necessary and sufficient condition which a property must satisfy, in order to define a set, is of great interest because it clarifies the relationship of axiomatic set theory to the paradoxes." Presumably, Gödel is referring to Axiom IV2 in *von Neumann 1925*, to the effect that a class is a set exactly when there is no surjection from that class onto the universe of sets. Gödel continues, "That this condition really gets at the essence of things is seen from the fact that it implies the axiom of choice, which formerly stood quite apart from other existential principles."[n] Gödel regards von Neumann's inferences as "not only very elegant, but also very interesting from the logical point of view." He continues, "Moreover I believe that only by going farther in *this* direction, i.e., in the direction opposite to constructivism, will the basic problems of abstract set theory be solved."

Ulam in letter 8, dated 6 January 1963, asks Gödel of his opinion of Paul Cohen's recent work on the independence of the continuum hypothesis, which introduced the forcing method. In a postscript, Ulam notes that he had in print expressed how he shared Gödel's belief in the falsity of CH, and inquires whether properties of sets of power $\aleph_1$ "seem now to be perhaps more interesting."

Gödel in his response, letter 9 dated 2 February 196[4], writes significantly that Cohen's work is a milestone "since it introduces for the first time a general method for independence proofs." He continues, "This is indispensable in set theory because the present axioms simply do not determine some of the most important properties of sets." These remarks gain more significance when it is pointed out that Gödel around 1942 had partial results toward the independence of the Axiom of Choice but neither pursued nor published this work.[o]

Gödel then voices agreement with Ulam about properties of sets of power $\aleph_1$ having become much more interesting, alludes to Hausdorff's "Pantachie Problem," and proceeds to describe it. This merits some discussion.

[m]Archives of the American Philosophical Society.

[n]Von Neumann's Axiom IV2 implies that there is a surjection F from the class ON of ordinals onto the class V of sets. But then, F induces a well-ordering $\prec$ of V defined by: $x \prec y$ exactly when the least member of the preimage $F^{-1}(\{x\})$ is less than the least member of the preimage $F^{-1}(\{y\})$. Hence, the Axiom of Choice in a strong, global sense holds. Also, by considering whether a class, once well-ordered by $\prec$, is bijective with ON or not, one sees that Axiom IV2 admits the following self-refinement: A class is a set exactly when there is no bijection from that class onto the universe of sets. This is how von Neumannn's approach to classes is most often characterized.

[o]For Gödel's partial results see *Moore 1988*, p. 130ff. and the correspondence with Rautenberg in this volume.

Most of *Hausdorff 1907* is devoted to the analysis of pantachies, and the last subsection is entitled "The Pantachie Problem." The term 'pantachie' derives from its initial use by *Du Bois Reymond 1880* to denote (everywhere) dense subsets of the real line and then to various notions connected with his work on rates of growth of real-valued functions and on infinitesimals. Discussing subsets of ${}^{\omega}\mathbb{R}$, the collection of (countable) sequences of reals, Hausdorff argued (p. 107) that "an infinite pantachie in the sense of Du Bois Reymond does not exist." Hausdorff then went on to redefine 'pantachie' as a subset of ${}^{\omega}\mathbb{R}$ maximal with respect to being linearly ordered under the eventual dominance ordering. (For $f, g \in {}^{\omega}\mathbb{R}$, *f eventually dominates g iff* $\exists m \in \omega \forall n \in \omega(m \leq n \rightarrow g(n) < f(n))$.) This anticipated Hausdorff's later work on maximal principles, principles equivalent to the Axiom of Choice. For an ordered set $\langle X, < \rangle$, an (ω_1, ω_1^*)*-gap* is a set $\{x_\alpha \mid \alpha < \omega_1\} \cup \{y_\alpha \mid \alpha < \omega_1\} \subseteq X$ such that $x_\alpha < x_\beta < y_\beta < y_\alpha$ for $\alpha < \beta < \omega_1$, yet there is no $z \in X$ such that $x_\alpha < z < y_\alpha$ for $\alpha < \omega_1$.

Hausdorff's subsection "The Pantachie Problem" lists several problems, the first of which (p. 151) is whether there is a pantachie (in his sense) with no (ω_1, ω_1^*)-gaps. What Gödel describes in letter 8 as "what Hausdorff calls the 'Pantachie Problem' " is actually the following variant of Hausdorff's "Scale Problem" (p. 152): whether there is a subset of ${}^{\omega}\omega$, the collection of (countable) sequences of natural numbers, cofinal in the eventual dominance ordering and of ordertype ω_1. This problem would be among the first adequately analyzed by the emerging method of forcing: Solovay observed in the mid-1960s that it is consistent to have such an ω_1 cofinal subset of ${}^{\omega}\omega$ with $2^{\aleph_0}$ arbitrarily large.[p] Then Stephen Hechler in his dissertation of 1967 (see *Hechler 1974*) would prove that it is consistent that there is no such ω_1 cofinal subset of ${}^{\omega}\omega$.[q]

It is a testament to Gödel's continuity of thought that in 1970 he would frame axioms (see these *Works*, vol. III, p. 405ff.) about the existence of certain collections of functions linearly ordered by eventual dominance, axioms that were intended to imply that CH is false in a specific way. Among his postulations was a pantachie with no (ω_1, ω_1^*)-gaps, the focus of Hausdorff's "The Pantachie Problem". Solovay in his commentary in these *Works*, vol. III, p. 409, provides in effect a proof that the existence of such a pantachie implies $2^{\aleph_0} = 2^{\aleph_1}$. This result actually appears in *Hausdorff 1907* (Theorem V, p. 128), which was presumably the route to Gödel's invocation (p. 421) of $2^{\aleph_0} \geq \aleph_2$.

[p]In modern parlance, Solovay had observed that the forcing for adjoining any number of random reals is ${}^{\omega}\omega$ bounding, i.e. any member of ${}^{\omega}\omega$ in the generic extension is eventually dominated by a member of ${}^{\omega}\omega$ in the ground model. The argument is given by Solovay in these *Works*, vol. III, §6.5 on p. 414.

[q]Hechler introduced a notion of forcing for adjoining a function in ${}^{\omega}\omega$ eventually dominating all ground model functions in ${}^{\omega}\omega$. With repeated applications he was able to establish the general assertion that if, in the sense of the ground model, κ and λ are cardinals of uncountable cofinality such that $2^{\aleph_0} \leq \kappa$ and $\lambda \leq \kappa$, then there is a cardinal-preserving generic extension in which $2^{\aleph_0} = \kappa$ and there is cofinal subset of ${}^{\omega}\omega$ of ordertype λ.

Returning to the correspondence, Ulam in his final letter of significance, letter 10 (dated 17 February 1966), discusses possibilities for adapting notions like Borel and projective set to subsets of a set E of power $\aleph_1$. This is in the spirit of his question, raised two years earlier in letter 8 (6 January 196[4]), of interesting properties of sets of power $\aleph_1$ independent of CH.

In retrospect, what is remarkable about this Ulam–Gödel correspondence is the lack of any mention at its end of two major developments in set theory in the 1960s, developments having to do with both Ulam's fruitful speculations from the 1930s and Gödel's interest in new axioms for set theory. The first development is Dana Scott's 1960 result (see *Scott 1961*) that if there is a measurable cardinal, then $V \neq L$, and the result of William Hanf and Alfred Tarski around the same time (see *Hanf 1964* and *Tarski 1962*) that a measurable cardinal κ is "much larger" than the least strongly inaccessible cardinal.[r] Measurable cardinals, as was written above, had emanated directly from Ulam's work on measure in *Ulam 1930*, and Gödel pointed out Scott's result in *Scott 1961* and referred to *Tarski 1962* in footnote 20 of his 1964 revision of his article *Gödel 1947* on the continuum problem (see these *Works*, vol. II, pp. 260–261). Moreover, measures had figured prominently in the early correspondence, and Solovay in 1966 (see *Solovay 1971*) had established the equiconsistency of the following two theories: (a) ZFC + "there is a measurable cardinal", and (b) ZFC + "there is a measure on $\mathcal{P}(\mathbb{R})$".[s] The second development left unmentioned in the correspondence is the investigation of determinacy, which was getting into full swing in the 1960s. The determinacy of infinite games is now a mainstream of set theory that brings together both descriptive set theory and large cardinals in the analysis of sets of reals.[t] Ulam had essentially asked in the 1930s, in the famous book of problems kept at the Scottish Cafe in Lwów (see *Mauldin 1981*, p. 113ff.), when a specific player has a winning strategy in a certain two-player infinite game. The related question of when such a game is *determined*, i.e. when there is a winning strategy for one or the other player, would become the focus of subsequent

[r]A *strongly inaccessible* cardinal is an uncountable regular cardinal κ such that whenever $\lambda < \kappa$, then $2^\lambda < \kappa$. The Hanf–Tarski argument shows that a measurable cardinal κ is in fact the κth strongly inaccessible cardinal and a fixed point in the cardinal sequence in various strong senses.

[s]See *Kanamori 1997*, §16.

[t]See *Kanamori 1997*, chapter 6.

investigations. Ulam's compatriots Jan Mycielski and Hugo Steinhaus proposed in *Mycielski and Steinhaus 1962* what is now known as the Axiom of Determinacy, and Mycielski himself would become a close colleague of Ulam's, but perhaps not until after Mycielski emigrated to Boulder, Colorado in 1969.[u] With infinite games becoming a rich paradigm for the articulation of dichotomies across the breadth of set theory, it is intriguing to consider what Gödel would have made of the developments in this direction and how he would have regarded determinacy axioms for set theory.

Akihiro Kanamori

A complete calendar of the correspondence with Ulam appears on p. 460 of this volume.

[u]It is noteworthy that in his contribution to the Ulam festschrift *Cooper 1989*, Mycielski does not mention determinacy.

1. Ulam to Gödel

Cambridge
Jan. 16, 1939

Dear Dr. Gödel.

I would be very grateful to you for a reprint of your paper in the Proceedings.[a] Will your full paper appear soon? If possible, please send me a copy of your Princeton lectures.[b]—I think the remark we made in Williamsburg: non-existence of any completely additive measure in the class of projective sets makes it imperative to change the problem of measure! Do you think it is worth publishing?—In connexion with a possibility of generalizing the problem of measure this question occurred to me: Let us consider the Boolean algebra of all subsets of the interval (0, 1) modulo countable sets. Can this algebra which has 2^c elements be mapped homomorphically on an algebra with only c elements?

[a]*Gödel 1938.*

[b]*Gödel 1940.*

I intend to write you soon on some subjects which we discussed in Williamsburg. Hoping that this letter reaches you (please send me your address in Notre-Dame) and expecting to hear from you soon I am

Yours sincerely

S. Ulam

2. Ulam to Gödel[a]

August 14, 1942

Dear Gödel:

I should like very much to have your opinion about an idea which seems to me to hold out some promise of establishing the existence of non-measureable (Lebesgue) projective sets from the assumption of the continuum hypothesis or even, perhaps, a weaker hypothesis—namely, that the power of the continuum is smaller than that of the first inaccessible aleph. It may be distinguished from your result in that it would hold, presumably without the independent axiom of yours about constructible sets. I enclose a few pages where this possibility is sketched.

Also, I wanted to ask you whether you ever published your results on the non-measurable projective sets or our joint remark that from the construction used by me in Fundamenta Mathematica[e], vol. 16,[b] it would follow, through the use of your method, that it is not possible to find any measure at all for the projective sets that would be zero for sets consisting of single points, countably additive and not identically zero.

Also, I should appreciate very much your comment on the following question: Luzin wrote in several papers and in his book rather vaguely

[a]On letterhead of the Department of Mathematics, 306 North Hall, The University of Wisconsin, Madison. Gödel appears to have numbered the first three paragraphs of this letter.

[b]*Ulam 1930.*

about "really" constructive Borel sets.[c] He formulated the problem of existence of such sets of the fourth class or higher classes. (Zero-class are open and closed sets. First-class, F_σ and G_δ sets, etc.) His lack of precision in formulating what constitutes the "really" constructive sets makes his problems mathematically meaningless. It seems to me that one way to define his notion is to try to define recursive sets of points on the line or recursive functions of the real variable as follows: Let us order, once for all, the sequence of intervals with rational endpoints $\omega_1, \omega_2, \ldots,$ $\omega_n, \ldots$. By a recursive open set we shall mean a set which can be represented in the form $Z = \sum_{k=1}^{\infty} \omega_{n_k}$ where the sequence of integers $n_1, n_2 \ldots$ $n_k \ldots$ is recursive in the accepted sense. A recursive closed set would be a complement of a recursive open set. A recursive G_δ-set would be one that can be written as $Z = \prod_l \sum_k \omega_{n_{k,l}}$, where the double sequence of integers $n_{k,l}$ is recursive. One can proceed in this fashion and define Borel sets that are recursive for any finite class and even some transfinite ordinal classes. Of course, one can start with various definitions of recursiveness for sequences or multiple sequences of integers.

The first problem that arises is this: Do there exist recursive Borel sets of the n-th ⟦class⟧ that do not belong to any lower class? An analogous question for recursive analytic or projective sets, if one defines a recursive analytical set as a projection on the line of a recursive plane G_δ-set. Do you have theorems in this direction? By the way, it seems that a possible definition of a recursive continuous (or more generally Baire) function of a real variable x would be this: For every rational interval the set $f^{-1}(\omega_n)$ is a recursive open (or Borel) set. [$f^{-1}(\omega_n)$ is the set of all x such that $f(x) \in \omega_n$.]

2 | I have studied the following situation which generalizes algebraically the idea of Boolean algebra and gives one posssibility, it seems to me, of a treatment of quantifiers on an algebraical basis: Suppose that we have a set E, its product with itself $E \times E = E^2$. (We single out a point p_0 in E and consider the subsets $p_0 \times E$ and $E \times p_0$ of E^2, as representing E itself.) Let us consider two given classes of subsets of E^2: $\mathfrak{A}$ ~~called~~ $\mathfrak{B}$ ~~and~~ . We shall call them projectively isomorphic if there exists a one-to-one transformation T of sets from $\mathfrak{A}$ to $\mathfrak{B}$ satisfying the following:

1. $T(A - B) = T(A) - T(B)$
2. $T(A \times B) = T(A) \times T(B)$
3. $T(\mathrm{Proj}_E(A)) = \mathrm{Proj}_E(T(A))$, (if all the sets on the left belong to the class $\mathfrak{A}$).

I have not completed any systematic investigation of such systems under the notion of projective isomorphism, but have noticed various facts about them. This set-up permits the making of a mathematical investigation of logic more completely, it seems to me, than the mere treatment through Boolean algebras. I studied in particular the question of closure of a finite number of sets under the Boolean operations and the operations of product and projection. The interesting feature of it is that one can obtain an infinite number of sets starting from only two sets given in E^2. Do you think that it is worthwhile undertaking a systematic investigation in this direction and if there is, perhaps, something in the logical literature with which I am not familiar, covering this already? As an example of theorems that I have obtained I may say that they arise from the fact that there exist systems which are isomorphic as Boolean algebras but not projectively isomorphic. The model on which I studied this situation is the case where E is a set of integers.—I enclose a few pages on which I sketched a possibility of obtaining constructive nonmeasurable sets.[d] For the understanding of it I may add that, as you know, it is sufficient to obtain $\aleph_1$ sets of measure zero ⟨whose sum would be the whole interval⟩ such that the system itself would be constructive, e.g. in the sense of Kuratowski's papers in Fundamenta M. vols. 27, 28, 29.[e] The property (w) ⟨used in the enclosed pages⟩ is the following: If a set Z contains a number x, it contains all numbers differing from it by a rational number.

I am extremely interested in knowing on what you are working now and to learn your results about the intuitionistic logic and the continuum hypothesis. Please write me if you find time.

[c] Luzin's book is *Luzin 1930*. In it Luzin discussed (pp. 89–104) the "constructive existence" of Borel sets in the first four classes. (The Borel sets are ramified into a hierarchy consisting of classes K_α for $\alpha < \omega_1$.) Luzin called (p. 89) 'canonical' those members of K_α given by "particularly simple" properties, stipulating that every member of K_α is to be a countable union of canonical members. In a footnote Luzin observed that the definition of 'canonical' was vague since when a property is "particularly simple" was not specified. Luzin went on to describe the known "arithmetical definitions" of sets in the first four classes. See also *Luzin 1930a* where Luzin states (p. 65) as a problem to demonstrate the "constructive existence" of a Baire function of class 5. The Baire hierarchy of real functions has a direct correlation with the Borel hierarchy of sets of reals.

[d] These pages are included in the *Nachlaß* as document number 012880. See also the introductory note.

[e] The relevant papers are *Kuratowski 1936, 1937, 1937a, 1937b* and *1937c*. *Kuratowski 1937* introduces the concept of an 'elementary' projective set, which is defined through relativization. In the document 012880 Ulam describes constructions similar to Kuratowski's, constructions that do not appeal to the Axiom of Choice. See also the introductory note.

With best regards and greetings,

Yours sincerely,

S. Ulam

3. Gödel to Ulam[a]

Dear Ulam

Excuse me please for not having ~~answered~~ ⟨replied earlier to⟩ your letter of Aug 14. The ~~first~~ sec. of your questions whether I publ. my results on non⟦-⟧meas. proj. set was I think answered by this very fact. As to your quest. about constr. ~~projective~~ ⟨Borel⟩ sets it seems to me that the univ. sets constr. in the usual manner are recursively proj. in your sense; ~~but~~ ⟨What⟩ Luzin ~~seems to have something else in~~ had in mind ~~is hard to see because I (really) don't know~~ when he considers the Baire-set ⟨of order 3⟩ as more constructive than the univ. sets. ~~I rea~~ is ~~hard to see~~ ⟨I don't know at all⟩ unless he meant something like "perspicuity" of the construction.—Your generalized Boole⟦an⟧ algebras, with the op. ×, proj would amount to a calculus of relations which is still
2 more restricted | than Schröder⟦'⟧s [which already does not allow to express all statements of the first order.] It would be interesting whether the problem of decision could be solved for this system. As to your ⟨interesting⟩ sketch[b] of a proof for the exist. of non⟦-⟧meas. proj. sets I am sorry that already the sec. step is not clear to me. Furthermore there is no reason why the proj. character should be preserved in sums of $\aleph_1$ summands. Finally it seems to me that if the arg went through it would give ~~a pro~~ ~~the constr.~~ non⟦-⟧measurable sets without the ax. of choice which is unlikely. Or do you have a diff. opinion about this question. ~~As for me~~ ⟨On ~~all~~ these grounds⟩ I am ~~more than~~ ⟨extremely⟩ doubtful about your whole scheme of proof.

With best reg. & greetings.

Your ——

[a] Undated letter draft.

[b] Gödel is referring to *Nachlaß* document 012880, enclosed with letter 2 by Ulam.

⟦Written vertically in the right margin:⟧ and also bec. I have a feeling that the measurability of all proj. sets is consist. log. with the ax. of choice.

4. Ulam to Gödel

Dec. 6, 1947

Dear Gödel,

Have just read your wonderful article in the Monthly on the Continuum hypothesis.[a] Would be very grateful to you for a reprint.

As you know, I believe in the falsity of the cont. hypothesis in the same sense as you state it in your predictions. i.e. that new axioms, as unavoidable as the axiom of choice will be added— from which it will follow that c is, for example the first inaccessible aleph. (I think that we discussed such possibilities in 1938 and later.)

May I indicate some more grounds for this feeling?

In a paper in Fundamenta[b] ⟨...Theorie der Mass. in allg. Mengenlehre⟩ (I think vol. 16 ⟨*or 15*⟩) I proved the impossibility of an absolutely additive measure function for *all* subsets of a set of power $\aleph_1$, then the same, without any hypotheses for sets of power $\aleph_2 \ldots$ all alephs less than the first inaccessible one. Now I believe that it *is* possible to have such a measure for the first inaccessible aleph, call it $\aleph_i'$. The reason is that probably one can "construct" all its subsets, somewhat as you construct "all" subsets of the continuum in your book in the Princeton series;[c] *and at the same time* construct a measure of the sets as one goes along with transfinite procedure. This means, in particular, the *im*possibility of a class of decompositions of $\aleph_i'$ with the properties given in my paper in Fund. 16. Of this, I am fairly certain.

Now, in addition, it seems very likely to me that some axiom, asserting the possibility | of existence of "more and more" sets definable by 2
transfinite induction can be postulated within the class of all subsets of the continuum, in which case it would be impossible to reject the conclusion that ⟦the⟧ continuum is $\aleph_i'$. ($\aleph_i^0 = \kappa^0$.)

[a] *Gödel 1947.*

[b] *Ulam 1930.*

[c] *Gödel 1940.*

At present I cannot find a sufficiently general and natural statement for an axiom of this sort.

I was writing you, for the last few months a longish letter on such set-theoretical questions but it is still unfinished.—Plan to be in Princeton towards the end of January and would like very much if I could see you and have a discussion on such problems.

As I think I mentioned to you once, I tried to examine properties of a class $\langle\mathcal{R}\rangle$ of sets (say of real numbers) with the properties:

(1.) $\mathcal{R}$ is a Borel field.

(2.) Given an arbitrary division of the interval I into disjoint sets, consisting of at least two elements each: $I = A_1 + A_2 + \ldots + A_\xi + \ldots$, then there exists in $\mathcal{R}$ a set having one element in common with *each* of the A_ξ: $(Z \epsilon \mathcal{R},\ Z \cdot A_\xi = \text{one pt.})$.

The first question was whether $\mathcal{R}$ must be the class of *all* subsets of I—up to countable sets. Now the axiom I had in mind would assert the possibility of constructing classes of sets—which ~~do not have this~~ satisfy 1, 2 and are *not* the full class, in fact of course more generally—any non-contradictory set of properties—and different from the class of *all* subsets. Somehow the axiom should perhaps assert a *freedom of construction* beyond that given by the axiom of choice—this still *within* a given system of axioms and not as a "meta-system" statement.

Please send me a reprint then: would be very grateful to you if you let me know whether you will be in Princeton towards end of January.

With cordial greetings,

Yours as ever

S. Ulam

5. Ulam to Gödel

October 25, 1957

Dear Gödel:

Many thanks for your letter and the copy of your letter to Price.[a]

[a] G. B. Price, professor of mathematics at the University of Kansas, was managing editor for the *Bulletin* for volume 64.

It was my intention, of course, to discuss, in an article on von Neumann, his work on logic, foundations, and set theory. This will probably occupy several pages. It would be wonderful, of course, if you were willing to write a special article on it since it would be much more authoritative and, to mention just one thing, I have always regarded as the principal value of von Neumann's articles on axiomatization, the fact that you used his system or a similar one for some of your work!

I was in Princeton for half a day a week ago, and dropped into your office, but you were not there. I hope to be in Princeton some time this winter and would like very much to have a chance again to continue our discussions of this spring. When I know more precisely the date, I will let you know in advance.

Cordially yours,

S. Ulam

P.S. I would like to have the chance of showing you the manuscript of the article on von Neumann, when I have it finished—probably within the next two months.

6. Gödel to Ulam

Princeton, Nov. 8, 1957

Dear Ulam:

Thank you for your letter of Oct. 25. You are doing me too much honour by seeing the principal value of von Neumann's papers on axiomatics of set theory in the fact that I used his, or a similar, system of axioms! His results, it is true, are mathematically not very intricate, but nevertheless, in my opinion, quite important for the foundations of set theory. In particular I believe that his necessary and sufficient condition which a property must satisfy, in order to define a set, is of great interest, because it clarifies the relationship of axiomatic set theory to the paradoxes. That this condition really gets at the essence of things is seen from the fact that it implies the axiom of choice, which formerly stood quite apart from other existential principles. The inferences, bordering ~~onn~~ the paradoxes, which are made possible by this way of looking at things, seem to me, not only very elegant, but also very interesting from the logical point of view. Moreover I believe that only by going farther in *this* direction, i.e., in the direction opposite to constructivism, will the basic problems of ~~set~~ abstract set theory be solved.

I don't know what von Neumann's attitude toward his work on the foundations of set theory was in his later years. Was he perhaps prejudiced against investigations of this kind because the consistency of the axioms cannot be proved?

Morgenstern told me that you may come to Princeton in the near future. It would be very nice if I could see you in that case.

Cordially yours,

Kurt Gödel

[1] and also by his general considerations about the actually occurring mathematical formalisms, although these form only a *very* small part of all possible formalisms

7. Gödel to Ulam[a]

Princeton, Jan. 28, 1958.

Dear Ulam:

I am sending you herewith the changes ⟨in⟩, and additions to, your report about v. Neumann's work on the the foundations of mathematics, which I would suggest. I also have reformulated my remark which you quote on p. 18. Incidentally I believe that this remark would better be quoted in connection with insertion 4 at the place marked by a star.

Concerning the relationship of v. Neumann's work to mine and Turing's I would like to add the following: ⟨To my knowledge⟩ the only passage in von Neumann's writings ~~I know of~~ which ⟨~~perhaps~~ ⟩ can be interpreted to be (as you rightly say) a "vague forecast" of the existence of undecidable propositions in any formal system (although certainly not of the unprovability of consistency) is the one you ~~quote~~ ⟨mention⟩ on p. 15. But this *one* ⟨doubtful⟩ passage (which, incidentally, has very little to do with the main purpose of the paper concerned) certainly does not justify what you say on p. 17, line 8 and line 17.

~~Of course~~ I do not want to deny that v. Neumann, by his virtuosity in using formalism,[1] ~~stimulated research [?]~~ [b] ~~clarify~~ ⟨~~on, and~~ contributed toward a clarification of,⟩ the concept and limits of ~~[?]~~ ⟨formalism⟩. However ~~I don't think~~ the words "anticipated" or "forerunner" are ~~suitable~~ ⟨much too strong⟩ for describing this⟨e⟩ situation.[2]

Thank you for sending me your collection of problems. I should be delighted to see you, if you came to Princeton sometime in the near future.

Sincerely yours

Kurt Gödel

[a] Retained copy.

[b] The words here are illegible, but may say "and attempted to".

[2] "prepares the grounds", which you say on p. 17, line 18, is a much more suitable expression.

8. Ulam to Gödel[a]

January 6, 1963[b]

Dear Gödel:

Last August I had to give a lecture, the so-called von Neumann lecture, which is given annually. The subject was "Combinatorial Analysis in Infinite Sets and Some Physical Theories."[c] In the first part of it I discussed your results on the continuum hypothesis and I also alluded to the results of Cohen, which I heard about but I have not seen his proof. I still do not know the details of it.

What do you think of these results? It seems to me that yours and Cohen's work really initiates, so to say, "non-Cantorian" set theories—really in analogy to non-euclidean geometries.

I am now writing up my lecture—it should be ready in a few days and I will permit myself to send you a preprint.

With best regards and greetings and wishes for a Happy New Year,

Yours, as ever,

S. M. Ulam.

[a] On letterhead of University of California, Los Alamos Scientific Laboratory, P.O. Box 1663, Los Alamos, New Mexico.

[b] Internal evidence shows the year should be 1964.

[c] *Ulam 1964*.

P.S. In my little book, "A Collection of Mathematical Problems",[d] published four years ago, I have written, in the Introduction on page one, that your impression was that "in a suitably large and 'free' axiomatic system for set theory, the continuum hypothesis is false. This feeling, based on indications provided by results on projective sets and the abstract theory of measure, has been shared by the author for many years."[e] I am glad that this proved to be correct. In general, results on properties of sets of power, $\aleph_1$, etc., which used no hypothesis whatsoever, seem now to be perhaps more interesting?

S. M. U.

[d] *Ulam 1960.*

[e] This entire quote was set off by Gödel in brackets.

9. Gödel to Ulam

February 10, 1964

Professor S. M. Ulam
Los Alamos Scientific Laboratory
P. O. Box 1663
Los Alamos, New Mexico

Dear Professor Ulam:

Many thanks for your letter of January 6.

A sketch of Cohen's proof of the independence of the continuum hypothesis has just come out in the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.[a] I think very highly of Cohen's work. It seems to me to mark a milestone in the history of set theory, since it introduces for the first time a general method for independence proofs. This is indispensable in set theory because the present axioms simply do not determine some of the most important properties of sets.

I perfectly agree with you that properties of subsets of power $\aleph_1$ have now become much more interesting. In particular what Hausdorff calls the "Pantachie Problem", i.e., whether there exist $\aleph_1$ sequences of integers whose orders of growth (Wachstumsordnungen) surpass any given

[a] *Cohen 1963* and *1964*.

order, seems to me a basic question about the structure of the continuum, once the continuum hypothesis is dropped.

I am looking forward to the preprint of your v. Neumann Lecture.

Sincerely yours,

Kurt Gödel

10. Ulam to Gödel[a]

February 17, 1964

Dear Professor Gödel:

Please find enclosed two copies of a lecture given in 1963. At the time it was given Paul Cohen's paper had not yet been published. I am sending *two* copies in case the library in the Institute or somebody else would want to have one.

It was very nice to see you at Morgenstern's last week and I only wish I could see you again soon—to discuss some of the many problems in set theory which now appear to me even more important. In particular, it seems to me, given a set E of power $\aleph_1$, it might be interesting to consider a class of subsets of it which would contain "elementary" subsets and also all sets obtained by, say, Borel operations on them. This in analogy to studying on the interval a class of sets which contains the elementary subsets, e.g., the subintervals, and then building new sets from them. The question is: what should we consider as an elementary subset of the set of all ordinals up to Ω? One could include among them all sets which form arithmetical progressions in this set (always including also limit ordinals of the sequence of such). Or perhaps one could consider as elementary the "recursive" subsets. Then if we add all countable sums of such to our class, complements, etc., that is to say the analogue of the Borel class, some questions arise. For example, can one define a countably additive measure function for all such sets? One could, of course, go farther and define projective operations on such sets by considering the direct products of the set E, again in analogy with the case of the continuum of real numbers. This would be a wider class of sets and I must say

[a] On letterhead of University of California, Los Alamos Scientific Laboratory, P.O. Box 1663, Los Alamos, New Mexico.

that I do not know whether there were studies of such in recent literature. The questions of existence of various "paradoxical" decompositions seem to me of interest. Can one consider the set E as a group in some natural way?

Needless to say, similar problems arise for $\aleph_2$, etc. Since it might be natural to assume that the continuum hypothesis is not true, I would think that such problems may be sensible. By the way, I would assume the usual axioms of set theory as used by you *and* the axiom of choice
2 but *not* the continuum hypothesis, | of course.

Perhaps I will be able to visit in Princeton sometime during the Spring and would like very much to be able to see you at that time.

With my very best regards and greetings,

Sincerely,

S. M. Ulam.