Introduction

The higher infinite refers to the lofty reaches of the infinite cardinalities of set theory as charted out by *large cardinal hypotheses*. These hypotheses posit cardinals that prescribe their own transcendence over smaller cardinals and provide a superstructure for the analysis of strong propositions. As such they are the rightful heirs to the two main legacies of Georg Cantor, founder of set theory: the extension of number into the infinite and the investigation of definable sets of reals. The investigation of large cardinal hypotheses is indeed a mainstream of modern set theory, and they have been found to play a crucial role in the study of definable sets of reals, in particular their Lebesgue measurability. Although formulated at various stages in the development of set theory and with different incentives, the hypotheses were found to form a *linear* hierarchy reaching up to an inconsistent extension of motivating concepts. All known set-theoretic propositions have been gauged in this hierarchy in terms of consistency strength, and the emerging structure of implications provides a remarkably rich, detailed and coherent picture of the strongest propositions of mathematics as embedded in set theory.

The first of a projected multi-volume series, this text provides a comprehensive account of the theory of large cardinals from its beginnings through the developments of the early 1970's and several of the direct outgrowths leading to the frontiers of current research. A further volume will round out the picture of those frontiers with a wide range of forcing consistency results and aspects of inner model theory. A genetic account through historical progression is adopted, both because it provides the most coherent exposition of the mathematics and because it holds the key to any epistemological concerns. With hindsight however the exposition is inevitably Whiggish, in that the consequential avenues are pursued and the most elegant or accessible expositions given. Each section is a modular unit, and later sections often describe how concepts discussed in earlier sections inspired the next advance. With speculations and open questions provided throughout, the reader should not only come to appreciate the scope and significance of the overall enterprise but also become prepared to pursue research in several specific areas.

In what follows a historical and conceptual overview is given, one that serves to embed the sections of the text into a larger framework. In an appendix larger and more discursive issues that may be raised by the investigation of large cardinals are taken up. See Hallett [84], Lavine [94], Moore [82], and Fraenkel–Bar-Hillel–Levy [73] for more on the development of set theory; several themes that are only broached here are substantiated in at least one of these sources.

The Beginnings of Set Theory

Set theory had its beginnings in the great 19th Century transformation of mathematics that featured the arithmetization of analysis and a new engagement with abstraction and generalization. Very much new mathematics growing out of old, the subject did not spring Athena-like from the head of Cantor but in a gradual process out of problems in mathematical analysis. In the wake of the founding of the calculus by Leibniz and Newton the function concept had been steadily extended from analytic expressions toward arbitrary correspondences, in the course of which the emphasis had shifted away from the continuum taken as a whole to its construal as a collection of points, the real numbers. The first major expansion had been inspired by the explorations of Euler and featured the infusion of infinite series methods and the analysis of physical phenomena, particularly the vibrating string.

Working out of this tradition the young Cantor in the early 1870's established uniqueness theorems for Fourier series in terms of their points of convergence, theorems based on collections of reals *defined* through a limit operation iterable into the infinite. In a crucial conceptual move Cantor began to investigate such collections and infinitary enumerations for their own sake, and this led first to basic concepts in the study of sets of reals and then to the formulation of the transfinite numbers. Set theory was born on that December 1873 day when Cantor established that the reals are uncountable, i.e. there is no one-to-one correspondence between the reals and the natural numbers, and in the next decades was to blossom through the prodigious progress made by him in the theory of ordinal and cardinal numbers. But a synthesis of the reals as representing the continuum and the new numbers as representing well-orderings eluded him: Cantor could not establish the Continuum Hypothesis, that the cardinality $2^{\aleph_0}$ of the set of reals is the least uncountable cardinality $\aleph_1$, part of his problem being that he could not define a well-ordering of the reals.

Cantor came to view the finite and the transfinite as all of a piece, similarly comprehendable within mathematics, and delimited by what he termed the "Absolute" which he associated mathematically with the class of all ordinals and metaphysically with God. As part of this realist picture Cantor viewed sets, at least until the early 1890's, as inherently structured with a well-ordering of their members. Ordinal and cardinal numbers resulted from successive abstraction, from a set x to its ordertype $\overline{x}$ and then to its cardinality $\overline{\overline{x}}$.

But such a structured view served to accentuate a growing stress among mathematicians, who were already exercised about two related issues: whether infinite collections can be investigated within mathematics at all and how far the function concept is to be extended. The entrenched positions being finitism and constructivism, there was open controversy after Ernst Zermelo [04] formulated what he soon called the Axiom of Choice and established his Well-Ordering Theorem, that the axiom implies every set can be well-ordered.

With axiomatization assuming a general methodological role in mathematics Zermelo [08a] soon published the first axiomatization of set theory. But as with Cantor's work the move was in response to mathematical pressure for a new context: Beyond the stated purpose of securing set theory from paradox Zermelo's main motive was apparently to buttress his Well-Ordering Theorem by making explicit its underlying set existence assumptions. In the process, he shifted the focus away from the transfinite numbers to a combinatorial view of sets structured solely by $\in$ and simple operations. Extracted from a specific proof (for the Well-Ordering Theorem in his [08]) Zermelo's axioms had the advantages of simplicity and open-endedness. The generative set formation axioms, especially Power Set, were to lead to Zermelo's [30] later adumbration of the cumulative hierarchy view of sets, and the vagueness of the *definit* property in the Separation Axiom was to invite Thoralf Skolem's [23] proposal to base it on first-order logic.

Skolem's move was in the wake of a mounting initiative, one that was to expand set theory with new viewpoints and techniques as well as to invest it with a larger foundational significance. Gottlob Frege is regarded as the greatest philosopher of logic since Aristotle for developing his 1879 *Begriffsschrift* (quantificational logic), establishing a logical foundation for arithmetic, and generally stimulating the *analytic tradition* in philosophy. The architect of that tradition is Bertrand Russell who in his early years, influenced by Frege and Giuseppe Peano, wanted to found all of mathematics on the certainty of logic. The vaulting expression of that ambition was the 1910-3 three volume *Principia Mathematica* by Alfred Whitehead and Russell. But Russell was exercised by his well-known paradox, one which led to the tottering of Frege's mature formal system. As a result *Principia* was encased in a complex logical system of different types and intensional predications ultimately breaking under his Axiom of Reducibility, a fearful symmetry imposed by an artful dodger.

It remained for David Hilbert to shift the ground and establish mathematical logic as a field of mathematics. Russell's philosophical disposition precluded his axiomatizing logic, but Hilbert brought it under scrutiny as he did Euclidean geometry by establishing an axiomatic context and raising the crucial questions of consistency and later, completeness. This largely syntactic approach was soon given a superstructure when in response to intuitionistic criticism by Luitzen Brouwer and Hermann Weyl, Hilbert developed proof theory and proposed his program of establishing the consistency of classical mathematics with his metamathematics. These issues gained currency because of Hilbert's preeminence, just as mathematics in the large was expanded by his reliance on non-constructive proofs and transcendental methods. Through this expansion the Axiom of Choice became a mathematical necessity, particularly because of maximality arguments in algebra, and arbitrary functions became implicitly accepted in the growing investigation of higher function spaces. With the increasing emphasis on frameworks and structures, the power set operation became incorporated into mathematics.

Throughout, Zermelian set theory grew as the mathematical repository of foundational concerns and initiatives. As the first result of his axiomatic set theory Zermelo [08a] himself put the Russell paradox argument to use to show that for any set x there is a set $y \subseteq x$ such that $y \notin x$ (so that there is no universal set). Friedrich Hartogs [15] in effect converted the Burali-Forti paradox into the existence for any set x of a well-orderable set y not injectible into x. Analyzing the Zermelo [08] proof Kazimierz Kuratowski [21] provided that definition of the ordered pair, antithetical to Russell's type-ridden theory, which became the standard way to reduce the theory of relations to sets. And then Skolem [23] made his proposal of rendering Zermelo's Separation Axiom in terms of properties expressible in first-order logic.

More than that, Skolem intended for set theory to be based on first-order logic with $\in$ construed syntactically and without a privileged interpretation. This becomes clear in his application of the Löwenheim-Skolem theorem to get the Skolem paradox: the existence of countable models of set theory although it entails the existence of uncountable sets. Ironically, Skolem intended by this means to deflate the possibility of set theory becoming a foundations for mathematics, but following Kurt Gödel's work Skolem's syntactical approach to set theory came to be accepted. And again the ways of paradox were absorbed into set theory, as the Löwenheim-Skolem theorem came to play an important internal role when semantic methods were ushered in by Alfred Tarski.

Skolem [23] also and Abraham Fraenkel [21, 22] independently proposed the addition of the Replacement Axiom to Zermelo's list, and this axiom soon figured in a counter-reformation of sorts. John von Neumann [23] introduced the ordinals (transitive sets well-ordered by $\in$) and showed that every well-ordering is isomorphic to an ordinal, thereby restoring Cantor's transfinite numbers *as* sets. No longer were the numbers abstractions, but in the new formulation became incorporated into the Zermelian framework of sets built up by $\in$ and simple operations. Von Neumann's particular approach to axiomatization fostered the liberal use of proper classes in set theory and brought Replacement into prominence through its role in definitions by transfinite recursion.

With these developments before him Zermelo [30] presented his final axiomatization of set theory, incorporating Replacement and also Foundation. This axiomatization was in second-order terms, allowed urelements, and eschewed the Axiom of Infinity, but shorn of these features it became the standard Zermelo-Fraenkel (ZFC) one when recast in the soon to emerge terms of first-order logic. The Foundation Axiom had been prefigured as a restricting possibility by Dmitry Mirimanov [17], Skolem [23], and von Neumann [25]. Zermelo offered a synthetic view of a succession of natural models for set theory, each a member of a next, essentially realizing that Foundation ranks the sets in these models into a cumulative hierarchy. In current terms the axiom stratifies the formal universe V of sets as $\bigcup_\alpha V_\alpha$, where V_0 is $\emptyset$, $V_{\alpha+1}$ is the power set of V_α, and V_δ for limit ordinals δ is the union of the V_α's for $\alpha < \delta$. In a notable inversion this *iterative conception* came to be accepted after Gödel's later advocacy as a heuristic for motivating the axioms of set theory generally, its open-endedness moreover promoting a principle of tolerance for motivating new hypotheses mediating toward Cantor's Absolute. Foundation is the one axiom unnecessary for the recasting of mathematics in set-theoretic terms, but it came to be the salient feature that distinguishes structural investigations specific to set theory. Indeed, it can be fairly said that modern set theory is the study of well-foundedness, the Cantorian well-ordering doctrines adapted to the Zermelian combinatorial conception of sets.

In the 1930's Gödel's incisive analyses brought about a transformation of mathematical logic based on new initiatives for mathematical elucidation. The main source was of course his Incompleteness Theorem [31], which led to the undecidability of validity for first-order logic and the development of recursion theory. But starting an undercurrent, the earlier Completeness Theorem [30] clarified the distinction between the semantics and syntax of first-order logic and secured its key instrumental property, compactness. Then Tarski [33, 35] set out his schematic definition of truth in set-theoretic terms, exercising philosophers to a surprising extent ever since. The groundwork had been laid for the development of model theory, and set theory was to be considerably enriched since the 1950's by model-theoretic techniques. First-order logic came to be accepted as the canonical language because of its semantic dexterity, Skolem's earlier suggestion for set theory taken up generally, and higher-order logics became downgraded as the workings of the power set operation in disguise.

So enriched and fortified by axioms, results, and techniques axiomatic set theory was launched on its independent course by Gödel's construction of L [38, 39] leading to the relative consistency of the Axiom of Choice and the Continuum Hypothesis. Synthesizing what came before, Gödel built on the von Neumann ordinals as sustained by Replacement to formulate a relative Zermelian universe of sets based on logical definability, a universe imbued with a Cantorian sense of order.

Large Cardinals

If the forgoing in brief (and with interpretative twists) is the high tradition of set theory from Cantor to Gödel, large cardinals are the trustees of older traditions in direct line from Cantor's original investigations of definable sets of reals and of the transfinite numbers. Before taking up the more continuous tradition having to do directly with the transfinite the other tradition is described, one that was to be revitalized in the 1960's by major new initiatives.

Descriptive set theory is the definability theory of the continuum, the study of the structural properties of definable sets of reals. In his most substantive approach to the Continuum Hypothesis Cantor had structured the problem via perfect sets and established that the closed sets have the *perfect set property* (11.3). Related were his contributions to measure theory, a theory that led to the Borel sets and of course to Lebesgue measure. The major incentives of descriptive set theory have been to approach sets of reals through definability as Cantor had done, and to investigate the extent of the *regularity properties*, of which Lebesgue measurability and Cantor's perfect set property are two. In a seminal paper Henri Lebesgue [05] provided the first hierarchy for the Borel sets and applied Cantor's diagonalization argument to show that the hierarchy is both proper and does not exhaust the definable sets of reals. The subject really began with Mikhail Suslin's discovery [17] of the *analytic* sets and fundamental results about this first level of the later *projective hierarchy.* The subsequent development by Nikolai Luzin, Wacław Sierpiński, and their collaborators featured *tree representations* of sets of reals, and it was through this opening that well-founded relations entered mathematical practice, the later tradition leading to Foundation and the iterative conception being quite separate and motivated by heuristics. The transfinite numbers, at least the countable ones, gained a further legitimacy through their necessary involvement in this work, contributing to the mathematical pressure for their general acceptance. Pressing upward in the projective hierarchy, by the early 1930's the descriptive set theorists had reached an impasse, one that was to be later explained by Gödel's delimitative results with L. (These matters are taken up in §§12, 13.)

The other, more primal Cantorian initiative, the mathematical investigation of the transfinite, was vigorously advanced into the higher infinite by Felix Hausdorff [08]. Dismissive of foundational issues, he pursued the structure of transfinite ordertypes for its own sake and was first to consider a large cardinal, a *weakly inaccessible* cardinal, as a natural limit point. Paul Mahlo [11, 12, 13] then studied stronger limit points, the *Mahlo* cardinals. Closure under the power set operation, intrinsic to the Zermelian set concept, was later incorporated in the concept of a *(strongly) inaccessible* cardinal by Sierpiński-Tarski [30] and Zermelo [30]. In the early semantic investigations before the general acceptance of first-order logic these cardinals provided the natural models for set theory, i.e. the corresponding initial segments of the cumulative hierarchy. (These topics are developed in §1.)

Measurability, the most prominent of all large cardinal hypotheses, embodied the first confluence of the Cantorian initiatives. Isolated by Stanisław Ulam [30] from measure-theoretic considerations related to Lebesgue measure, the concept *also* entailed inaccessibility in the transfinite. Moreover, the initial airing generated an open problem that was to keep the spark of large cardinals alight for the next three decades: *Can the least inaccessible cardinal be measurable?* (Measurability is discussed in §2.)

The further development of the higher infinite was to depend on model-theoretic techniques brought into set theory in the course of its larger development. Gödel's L was the first example of an *inner model*, a class (definable by a formula of first-order logic) including all the ordinals, which with $\in$ restricted to it is a model of the axioms. Gödel with L had in fact established the minimum possibility for the set-theoretic universe, and large cardinals were to provide the counterweight first in reaction and then for generalization. Gödel's realist speculations, especially about Cantor's Continuum Problem, contained the seeds of later heuristic arguments for large cardinal hypotheses:

The set-theoretic universe V viewed as the cumulative hierarchy $\bigcup_\alpha V_\alpha$ is open-ended and under-determined by the set-theoretic axioms, and invites further postulations based on *reflection* and *generalization.* In 1946 remarks Gödel [90: 151] suggested reflection in terms of a set-theoretic proposition being provable in "the next higher system above set theory", which proof being replaceable by one from "an axiom of infinity". This ties in with V cast as Cantor's Absolute being mathematically incomprehendable, so that any property ascribable to it must already hold in some sufficiently large V_α, some properties leading directly to large cardinal hypotheses. In a 1966 footnote Gödel [90: 260ff] acknowledged "strong axioms of infinity of an entirely new kind", generalizations of properties of $\aleph_0$ "supported by strong arguments from analogy". This ties in with Cantor's unitary view of the finite and transfinite, with properties like inaccessibility and measurability technically satisfied by $\aleph_0$ being too accidental were they not also ascribable to higher cardinals. Both reflection and generalization are latent in the eternal return of successive domains as envisioned by Zermelo [30]. Whatever the heuristics, the theory of large cardinals like other mathematical investigations was to be driven by open problems and growing structural elucidations. (These matters are taken up in §3. Other heuristic arguments are described in Maddy [88, 88a].)

The generalization of first-order logic allowing infinitary logical operations was to lead to the solution of that problem of whether the least inaccessible cardinal can be measurable. Tarski [62] defined the *strongly compact* and *weakly compact* cardinals by ascribing natural generalizations of the key compactness property of first-order logic to the corresponding infinitary languages. A strongly compact cardinal is measurable, and a measurable cardinal is weakly compact. Tarski's student William Hanf [64] then established (4.7) that *there are many inaccessible cardinals (and Mahlo cardinals) below a weakly compact cardinal.* In particular, *the least inaccessible cardinal is not measurable.* Hanf's work radically altered size intuitions about properties coming to be understood in terms of large cardinals. (These topics are developed in §4.)

In the early 1960's set theory was veritably transformed by structural initiatives based on new possibilities for constructing well-founded models and establishing relative consistency results. This was due largely to the invention of *forcing* by Paul Cohen [63, 64], who happened upon a remarkably fertile technique for producing extensions of models of set theory. In a different vein, a seminal result of Dana Scott [61] stimulated the investigation of *elementary embeddings* of inner models. The ultraproduct construction of model theory was just gaining currency when Scott took an ultrapower of V itself to establish (5.5) that *if there is a measurable cardinal, then* $V \neq L$. Large cardinal hypotheses thus assumed a new significance as a means for maximizing possibilities away from Gödel's delimitative construction. And Cantor's Absolute notwithstanding, Scott's construction began the liberal use of manipulative inner model constructions in set theory. It was in this richer setting that measurable cardinals came to play a central structural role, being necessary for securing well-founded ultrapowers (see 5.6 and before): *There is an elementary embedding* $j\colon V \to M$ *for some inner model* M *iff there is a measurable cardinal.* (These matters are taken up in §5.)

With reflection arguments emerging in the model-theoretic approaches taken in set theory, Azriel Levy [60a] established their broader significance and the close involvement of Mahlo cardinals. Then Hanf-Scott [61] formulated the *indescribable* cardinals, directly positing reflection properties in terms of higher-order languages, and showed that these cardinals provide a schematic approach to comparing large cardinals by size. Levy [71] then provided a systematic analysis, features of which were to occur in later contexts. (Indescribability is described in §6.)

Scott's result that if there is a measurable cardinal then $V \neq L$ naturally led to refinements both weakening the hypothesis and strengthening the conclusion. Notably, the first moves were made in the context of the infinitary combinatorics then being developed by Paul Erdös and his collaborators, the study of *partition* properties, which are transfinite generalizations of a result of Frank Ramsey [30]. Frederick Rowbottom [64, 71] then established a partition property for measurable cardinals (7.17), and using model-theoretic methods showed that such properties already imply that there are only countably many reals in L (8.3). This blending of model theory and infinitary combinatorics led to a spectrum of large cardinals positing strong versions of the Löwenheim-Skolem theorem, the *Rowbottom* and *Jónsson* cardinals in particular generating intriguing questions. Weaving in the crucial model-theoretic concept of a *set of indiscernibles* Jack Silver [66, 71] then analyzed what came to be regarded as the essence of transcendence over L, encapsulated by him and Robert Solovay [67] as a set $0^{\#}$ of integers coding a collection of sentences uniquely specified by indiscernibility conditions. Beyond a web of implications encircling the merely negative conclusion $V \neq L$, the existence of $0^{\#}$ is a strikingly informative assertion about just how starkly L is generated in a transcendent V. Subsequent results have buttressed the existence of $0^{\#}$ as a pivotal hypothesis, and its isolation is the first real triumph for large cardinals in the elucidation of set-theoretic structure. (These matters are taken up in Chapter 2.)

Returning to the early 1960's, if Gödel's construction of L had launched axiomatic set theory as a distinctive field of mathematics, then Cohen's technique of forcing began its transformation into a modern, sophisticated one. Starting with his work on the Continuum Hypothesis many problems that had been left unresolved were shown to be independent, as set theorists were presented a remarkably general and flexible scheme with intuitive underpinnings for constructing models of set theory. The thrust of research gradually deflated the Cantor-Gödel realist view with an onrush of new models, and shedding some of its foundational burden set theory became an intriguing mathematical subject where formalized versions of truth and consistency became matters for combinatorial manipulation as in algebra. From Skolem relativism to Cohen relativism the role of set theory for mathematics became even more evidently one of an open-ended framework rather than an elucidating foundation. From this point of view, that the ZFC axioms do not determine the cardinality $2^{\aleph_0}$ of the set of reals seems an entirely satisfactory state of affairs. With the richness of possibility for arbitrary reals and mappings, no axioms that do not directly impose structure from above should constrain a set as open-ended as the collection of reals or its various possibilities for well-ordering.

Inaccessible cardinals figured from the beginning in this sea-change, first in the concept of the *Levy collapse* and then in its use in Solovay's inspiring result [65b, 70] that *if there is an inaccessible cardinal, then in a submodel of a forcing extension every set of reals is Lebesgue measurable and has the perfect set property.* (The Axiom of Choice necessarily fails in this submodel.) As Cohen's independence of the Continuum Hypothesis did for the transfinite, this result on the regularity of sets of reals not only resolved old axiomatic issues but reinvigorated the Cantorian initiatives by suggesting new mathematical possibilities. Solovay [69] soon applied the ideas of his proof to show that measurable cardinals directly imply the regularity properties at the level of Gödel's delimitative results with L, revitalizing the classical program of descriptive set theory. Then Donald Martin and Solovay (cf. their [69]) applied large cardinal hypotheses at the level of $0^{\#}$ to push forward the old tree representation ideas, with the hypotheses cast in the new role of securing well-foundedness in this context. (These matters are taken up in Chapter 3.)

The perfect set property led to the first instance of a new phenomenon in set theory: the derivation of *equiconsistency* results based on the complementary methods of forcing and inner models. A large cardinal hypothesis is typically transformed into a proposition about sets of reals by forcing that "collapses" the cardinal to $\aleph_1$ or "enlarges" the power of the continuum to the cardinal. Conversely, the proposition entails the same large cardinal hypothesis in the clarity of an inner model. Solovay's result provided the forcing direction from an inaccessible cardinal to the proposition that every set of reals has the perfect set property. But Ernst Specker [57] had in effect established that if every set of reals has the perfect set property (and $\aleph_1$ is regular), then $\aleph_1$ is inaccessible in L (11.6). Thus, Solovay's use of an inaccessible cardinal was necessary, and its collapse to $\aleph_1$ complemented Specker's observation. Years later, Saharon Shelah [84] was able to establish the necessity of Solovay's inaccessible also for the proposition that every set of reals is Lebesgue measurable.

The emergence of such equiconsistency results is a subtle vindication of earlier hopes of Gödel: Propositions can indeed be resolved if there are enough ordinals, how many being specified by positing a large cardinal. But the resolution is in terms of the Hilbertian concept of consistency, the methods of forcing and inner models being the operative modes of argument. In a new synthesis of the two Cantorian initiatives, hypotheses of length concerning the extent of the transfinite are correlated with hypotheses of width concerning sets of reals. There is a telling antecedent in the result of Gerhard Gentzen [36, 43] that the consistency strength of arithmetic can be exactly gauged by an ordinal ε_0, i.e. transfinite induction up to that ordinal in a formal system of notations. Although Hilbert's program of establishing consistency by finitary means cannot be realized, Gentzen provided an exact analysis in terms of ordinal length. Proof theory blossomed in the 1960's with the analysis of other theories in terms of such lengths, the proof theoretic ordinals.

In the late 1960's a wide-ranging investigation of measurability was carried out with forcing and inner models. These developments not only provided an illuminating structural analysis, but suggested new questions and provided paradigms for the subsequent investigation of stronger hypotheses. Solovay [66, 71] brought the concept of *saturated ideal* to the forefront, establishing an equiconsistency result about *real-valued measurability*. Subsequent work showed that saturated ideals are a flexible generalization of measurability that can occur low in the cumulative hierarchy. Exploiting the technique of *iterated ultrapowers* developed by Haim Gaifman [64], Kenneth Kunen [70] established the main structure theorems for *inner models of measurability*. Not only do these models have the minimal structure of Gödel's L, but they turn out to be exactly the ultrapowers of each other, and such coherence amounts to strong evidence for the consistency of the concept of measurability. Kunen also established a characterization of the existence of $0^{\#}$ in terms of the non-rigidity of L: *$0^{\#}$ exists iff there is an elementary embedding $j\colon L \to L$.* Solovay isolated a set $0^{\dagger}$ that plays an analogous role for inner models of measurability that $0^{\#}$ does for L, and its existence has a similar characterization in terms of non-rigidity. (These topics are developed in Chapter 4.)

Even as measurability was being methodically investigated, Solovay and William Reinhardt were charting out stronger hypotheses. Taking the concept of elementary embedding as basic they independently formulated the concept of *supercompact* cardinal as a generalization of both measurability and strong compactness, and Reinhardt formulated the stronger concept of *extendible* cardinal with motivating ideas based directly on reflection. Reinhardt briefly considered an ultimate reflection property along these lines, but in a dramatic turn of events Kunen [71b] established that this *prima facie* extension is inconsistent: *There is no elementary embedding $j\colon V \to V$.* Kunen's argument turned on what seemed to be a combinatorial contingency, but his particular formulation has stood as the upper bound for large cardinal hypotheses. The initial guiding ideas shaped and delimited by a mathematical result, hypotheses just on the verge of this inconsistency were subsequently analyzed, as well as the weaker *n-huge* cardinals and *Vopěnka's Principle* to chart the terrain down to the extendible cardinals. The supercompact cardinals in particular became prominent as a source of new combinatorics and relative consistency results. Also, when refinements of elementary embedding in the form of *extenders* were formulated, weakenings of supercompactness in the form of *strong, Woodin,* and *superstrong* cardinals came to play crucial roles in later developments. (These topics are developed in Chapter 5.)

With this charting out of the higher infinite, the extensive research through the 1970's and 1980's considerably strengthened the view that the emerging hierarchy of large cardinals provides *the* measuring rod of exhaustive principles against which all possible consistency strengths can be gauged. First, the various hypotheses though arising from diverse motivations and historical happenstance nonetheless form a *linear* hierarchy, one neatly delimited by Kunen's inconsistency result. Typically for two large cardinal hypotheses, below a cardinal satisfying one there are many cardinals satisfying the other, in a sense prescribed by the first. Moreover, the weaker hypotheses through strong forms of measurability have been bolstered by a variety of equiconsistency results involving combinatorial propositions low in the cumulative hierarchy. In this respect, particularly intriguing is the work on the *Singular Cardinals Problem*, which showed that something as basic as rendering 2^κ large for singular strong limit cardinals κ essentially requires large cardinals. Finally, a variety of strong propositions have been informatively bracketed in consistency strength between two large cardinal hypotheses: The stronger hypothesis implies that there is a forcing extension in which the proposition obtains; and if the proposition obtains, there is an inner model satisfying the weaker hypothesis. Supercompactness has often figured as the upper bound, but sometimes n-hugeness and even the hypotheses just short of Kunen's inconsistency have played this role. (This wide-ranging exploration is the subject of volume II.)

If set theory serves as an open-ended framework for mathematics, as an autonomous field of mathematics it has become a remarkably successful investigation of well-foundedness, in large measure because large cardinals have been found to provide an elegant and fully sufficient superstructure for the study of consistency strength.

Determinacy

One of the great successes for large cardinals has to do with perhaps the most distinctive and intriguing development in modern set theory. Although the *determinacy of games* has roots as far back as Zermelo [13], the concept for infinite games only began to be seriously explored in the 1960's when it was realized that it led to the regularity properties for sets of reals. Jan Mycielski and Hugo Steinhaus in their [62] proposed the *Axiom of Determinacy*, at least for some inner model since it contradicts the Axiom of Choice. Then in 1967 Solovay made an initial connection with large cardinals and David Blackwell [67] with methods of descriptive set theory. Investigating further consequences of determinacy, fine mathematicians like Solovay, Martin, Yiannis Moschovakis, Kunen, and Alexander Kechris soon established an elaborate web of connections in the unabashed pursuit of structure for its own sake. Determinacy hypotheses seemed to settle many questions and provide new modes of argument, leading to an opaque realization of the old Cantorian initiatives concerning sets of reals and the transfinite with determinacy replacing well-ordering as the animating principle. By the late 1970's a more or less complete theory for the projective sets was in place, and with this completion of a main project of descriptive set theory attention began to shift to questions of overall consistency.

Martin [70] had early on shown that the existence of a measurable cardinal implies the determinacy of games for analytic sets, and through the 1970's he established results equating many measurable cardinals with levels of a difference hierarchy for analytic sets and then showed that a large cardinal hypothesis near Kunen's inconsistency implied determinacy at the next projective level. Then in the mid-1980's Matthew Foreman, Menachem Magidor, and Shelah made a major breakthrough about strong large cardinal hypotheses, and although not directly involving determinacy Martin, John Steel, and Hugh Woodin were able to build on this to establish the consistency of the Axiom of Determinacy relative to large cardinals. Woodin in fact established that *the Axiom of Determinacy is equiconsistent with the existence of infinitely many Woodin cardinals*, pinpointing the axiom in consistency strength above measurable cardinals but far below supercompact cardinals. This unifying result was a resounding triumph for the modern methods of set theory and an unexpected affirmation of the relevance of large cardinals. Woodin's subsequent results about other determinacy hypotheses and infinite combinatorics speak to the great progress that has been made and the promise of deeper insights to come. (These matters are taken up in Chapter 6.)