F15

1. [30 Points] Consider the system of differential equations

$$Y' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} Y.$$

   1. Find the eigenvalues associated to this system.
   2. Find the general solution of this system.
   3. Sketch the phase plane for this system.
   4. Find the solution $Y(t)$ of this system that satisfies $Y(0) = (0, 1)$.

2. [15 Points] Consider the family of systems of diferential equations given by

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & A \end{pmatrix} Y$$

where A is a parameter. Find **all** values of A for which this system has different types of phase planes, e.g., saddles, spiral sinks, repeated eigenvalues, etc. Then draw the curve corresponding to these A-values in the trace-determinant plane.

3. [20 Points]

   1. Write the second order differential equation for the damped harmonic oscillator with mass $m = 1$, spring constant $k > 0$, and damping constant $b \geq 0$.
   2. Write this equation as a linear system. Be sure that you have written this system correctly; otherwise, all other answers will be wrong and you will receive no credit for this problem.
   3. What are the eigenvalues of this matrix?
   4. For which values of b and k does this system have complex eigenvalues?
   5. Repeated eigenvalues?
   6. For which values of b and k does this system have a saddle at the origin?

4. [15 Points] Suppose I give you a system of differential equations $Y' = AY$ for which the characteristic equation of A is $\lambda^2 - \lambda + 1 = 0$. Suppose I also tell you that the vector field at $(1, 0)$ is given by $(1, -1)$.

   1. Please give me a sketch of the solution curve in the phase plane satisfying $Y(0) = (1, 0)$.
   2. Also give me a sketch of the $x(t)$ and $y(t)$ graphs of this solution.

5. [20 Points] Quickies: answers only.

   1. Sketch the phase plane for the system
   $$Y' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} Y.$$
   2. Find all equilibrium points for the system
   $$Y' = \begin{pmatrix} a & a \\ b & b \end{pmatrix} Y$$
   where a and b are parameters.
   3. Find the general solution of the second order equation $y'' = 0$.
   4. Find all eigenvalues for the matrix
   $$\begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix}.$$