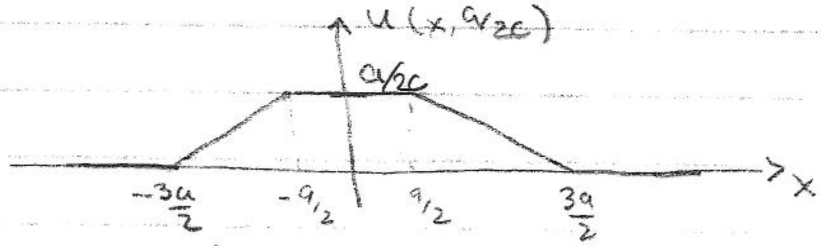
p.37 #5: From D'Alembert's formula,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \chi_{[-a,a]}(z)\,dz$$

Where $\chi_{[-a,a]}(z) = \begin{cases} 1 & \text{if } |z| \le a \\ 0 & \text{otherwise.} \end{cases}$

$$u(x, a/2c) = \frac{1}{2c}\int_{x-\frac{a}{2}}^{x+a/2} \chi_{[-a,a]}(z)\,dz.$$

$$= \frac{1}{2c} \times \text{length}\,(x - a/2,\ x + a/2) \cap [-a,a]$$



If one repeats this calculation for other values of t one finds the following sketches:

p.37 #10. Introduce new independent variables

$$\xi = ax + bt, \quad \eta = cx + dt$$

Then if $u(x,t) = \tilde{u}(\xi(x,t), \eta(x,t))$ we have

$$u_t = \tilde{u}_\xi \frac{\partial \xi}{\partial t} + \tilde{u}_\eta \frac{\partial \eta}{\partial t} = b\,\tilde{u}_\xi + d\,\tilde{u}_\eta$$

in like fashion

$$u_{tt} = b^2\,\tilde{u}_{\xi\xi} + d^2\,\tilde{u}_{\eta\eta} + 2bd\,\tilde{u}_{\xi\eta}$$

$$u_{tx} = ab\,\tilde{u}_{\xi\xi} + (bc + ad)\,\tilde{u}_{\xi\eta} + cd\,\tilde{u}_{\eta\eta}$$

$$u_{xx} = a^2\,\tilde{u}_{\xi\xi} + c^2\,\tilde{u}_{\eta\eta} + 2ac\,\tilde{u}_{\xi\eta}$$

Therefore $u_{xx} + u_{xt} - 20\,u_{tt} = (a^2 + ab - 20b^2)\,\tilde{u}_{\xi\xi} + (c^2 + cd - 20d^2)\,\tilde{u}_{\eta\eta}$

$$+ (2ac + bc + ad - 40bd)\,\tilde{u}_{\xi\eta}$$

Note that if we choose $a = 4, b = 1, c = 5, d = -1$, we have

$$u_{xx} + u_{xt} - 20\,u_{tt} = 81\,\tilde{u}_{\xi\eta} = 0$$

$\therefore$ $\tilde{u}_{\xi\eta} = 0 \Rightarrow \tilde{u}(\xi, \eta) = f(\xi) + g(\eta)$

just as in the derivation of D'Alembert's formula. Reverting to (x,t) variables we find

$$u(x,t) = f(4x + t) + g(5x - t).$$

.37 #10 (cont.) Now we solve for f & g in terms of the i.c.'s.

$$u(x,0) = \varphi(x) = f(4x) + g(5x)$$

$$u_t(x,0) = \psi(x) = f'(4x) - g'(5x)$$

Integrating the second equation gives

$$\tfrac{1}{4} f(4x) - \tfrac{1}{5} g(5x) = \tfrac{1}{4} f(0) - \tfrac{1}{5} g(0) + \int_0^x \psi(z)\,dz$$

Solving for f & g we find

$$f(4x) = \left(-\tfrac{20}{9}\right)\left(-\tfrac{1}{5}\varphi(x) - \tfrac{1}{4} f(0) + \tfrac{1}{5} g(0) - \int_0^x \psi(z)\,dz\right)$$

$$g(5x) = \left(-\tfrac{20}{9}\right)\left(-\tfrac{1}{4}\varphi(x) + \tfrac{1}{4} f(0) - \tfrac{1}{5} g(0) + \int_0^x \psi(z)\,dz\right)$$

Recombining f & g we find

$$u(x,t) = f(4x+t) + g(5x-t)$$

$$= \tfrac{4}{9}\varphi\left(x + \tfrac{t}{4}\right) + \tfrac{20}{9}\int_0^{x+\frac{t}{4}} \psi(z)\,dz$$

$$+ \tfrac{5}{9}\varphi\left(x - \tfrac{t}{5}\right) - \tfrac{20}{9}\int_0^{x-t/5} \psi(z)\,dz$$

$$= \tfrac{4}{9}\varphi\left(x + \tfrac{t}{4}\right) + \tfrac{5}{9}\left(x - t/5\right)$$

$$+ \left(\tfrac{20}{9}\right)\int_{x-t/5}^{x+t/4} \psi(z)\,dz$$

page 40 #3. Suppose $u(x,t)$ solves the wave equation

(a) Let $w(x,t) = u(x-y,t)$. Then $\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x-y,t)$ while $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}(x-y,t)$ so

$$\frac{\partial^2 w}{\partial t^2}(x,t) - c^2\frac{\partial^2 w}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x-y,t) - c^2\frac{\partial^2 u}{\partial x^2}(x-y,t) = 0.$$

(b) Let $w(x,t) = \frac{\partial u}{\partial x}(x,t)$. Then $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right)$ (Assuming that u is smooth enough that Clairaut's theorem applies and we can interchange the order of t & x derivatives.) Thus,

$$\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= \frac{\partial}{\partial x}\left\{\left(\frac{\partial^2 u}{\partial t^2}\right) - c^2\left(\frac{\partial^2 u}{\partial x^2}\right)\right\} = 0.$$

(c) Let $w(x,t) = u(ax,at)$. Then by the chain rule

$\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 u}{\partial t^2}$ & $\frac{\partial^2 w}{\partial x^2} = a^2\frac{\partial^2 u}{\partial x^2}$ so

$$\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} = a^2\frac{\partial^2 u}{\partial t^2} - a^2c^2\frac{\partial^2 u}{\partial x^2} = a^2\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\right) = 0.$$

p. 40 #6

(a) If $u(r,t) = \alpha(r) f(t-\beta(r))$ then

$$u_r = \alpha'(r) f(t-\beta(r)) - \alpha(r) f'(t-\beta(r))\beta'(r)$$
$$u_{rr} = \alpha''(r) f(t-\beta(r)) - 2\alpha'(r)\beta'(r) f'(t-\beta(r)) + \alpha(r) f''(t-\beta(r))(\beta'(r))^2 - \alpha(r) f'(t-\beta(r))\beta''(r)$$

while

$$u_t = \alpha(r) f'(t-\beta(r)), \quad u_{tt} = \alpha(r) f''(t-\beta(r))$$

$\therefore$ $u_{tt} = c^2\left(u_{rr} + \frac{(n-1)}{r} u_r\right) \Longleftrightarrow$

$$\alpha(r) f''(t-\beta(r)) = c^2\left(\alpha'' f - 2\alpha'\beta' f' + \alpha f''(\beta')^2 - \alpha\beta'' f' + \frac{(n-1)}{r}\left(\alpha' f - \alpha f'\beta'\right)\right)$$

Rearranging terms we find

$$[\alpha - c^2\alpha(\beta')^2] f'' + c^2\left\{\frac{n-1}{r}\alpha\beta' + \alpha\beta'' + 2\alpha'\beta'\right\} f' - c^2\left(\alpha'' + \frac{n-1}{r}\alpha'\right) f = 0$$

(b) If this equation is to be satisfied for any function f the coefficients of each term must vanish:

$$[\alpha - c^2\alpha(\beta')^2] = 0 \Rightarrow \beta'(r) = \pm 1/c$$
$$\Rightarrow \beta(r) = \pm \frac{1}{c} r + k_1$$

Also $\beta'' = 0$.

Considering the coefficient of f' (and using the fact that $\beta'' = 0$ gives

$$\left(\frac{n-1}{r}\right)\alpha\beta' + 2\alpha'\beta' = 0 \implies \frac{\alpha'(r)}{\alpha(r)} = -\frac{1}{2}\left(\frac{n-1}{r}\right)$$

$$\implies \begin{cases} \alpha(r) = k_2 & (\text{if } n=1) \\ \alpha(r) = k_2 r^{-\left(\frac{n-1}{2}\right)} & (\text{if } n>1) \end{cases}$$

Finally we have $\alpha''(r) + \left(\frac{n-1}{r}\right)\alpha'(r) = 0$. Clearly the formula for $n=1$ satisfies this equation. For $n>1$ we have $\alpha' = -\frac{1}{2}(n-1)\,k_2\, r^{-\left(\frac{n+1}{2}\right)}$ & $\alpha'' = \frac{1}{4}(n^2-1)\,k_2\, r^{-\left(\frac{n+3}{2}\right)}$.
Inserting these expressions into $\alpha'' + \left(\frac{n-1}{r}\right)\alpha' = 0$ implies

$$\frac{1}{4}(n^2-1) - \frac{1}{2}(n-1)^2 = 0$$

which holds only for $\boxed{n=3}$ (if $n>1$.)

$\therefore$ We have solutions only for $n=1$ & $n=3$.

(d) As we showed above, for $n=1$, $\alpha(r) = \text{const}$. So there is no attenuation, but if $n=3$,

$$\alpha(r) = k_2 r^{-1} = \frac{k_2}{r},$$

so the amplitude of the wave decreases inversely with the distance from the origin.

Page 44 #1: Using the techniques for ~~___~~ finding maxima and minima from calculus ~~__~~ one can show that the max. of $u(x,t)$ occurs at the origin and the minimum at $(1,T)$, consistent with the maximum principle.

Page 44 #2:

(a) If we consider the max of u over a rectangle $R(T_2) = \{(x,t) \mid 0 \le x \le l, 0 \le t \le T_2\}$ & a rectangle $R(T_1)$ with $T_2 > T_1$ then every point in $R(T_1)$ is contained