Homework #3 Solutions

p.45 #5 (a) If $u(x,t) = -2xt - x^2$ then

$u_t = -2x$ while $u_x = -2t - 2x$ and $u_{xx} = -2$

$$\therefore\ u_t = x u_{xx}$$

By elementary calculus one finds that the max. value of $u(x,t)$ is 1 which occurs at $(x,t) = (-1,1)$. This is not on the bottom or sides of the rectangle & hence the max. principle fails.

(b) If one tries to define $v_\epsilon(x,t) = u(x,t) + \epsilon x^2$ as in the proof of the max. theorem then

$$\partial_t v_\epsilon - x \partial_x^2 v_\epsilon = -2\epsilon x.$$

which is no longer strictly negative. Thus, we cannot rule out that v (& hence u) could have a max. inside or on the top of the rectangle.

p. 45 #8 Define

$$\mathcal{E}[u](t) = \int_0^L u^2(x,t)\,dx$$

Then

$$\frac{d\mathcal{E}}{dt} = 2\int_0^L u\,u_t\,dx$$

$$= 2k\int_0^L u\,u_{xx}\,dx$$

Int. by parts

$$= 2k\left\{ u u_x \Big|_0^L - \int_0^L (u_x(x,t))^2\,dx \right\}$$

use b.c.'s to replace $u_x(0,t)$ and $u_x(L,t)$

$$= 2k\left\{ -a_\ell (u(L,t))^2 - a_0 (u(0,t))^2 - \int_0^L (u_x(x,t))^2\,dx \right\}$$

Since a_ℓ and a_0 are positive, each term in braces is less than or equal to zero so

$$\frac{d\mathcal{E}}{dt} \le 0.$$

page 52 #16:

$$u_t - k u_{xx} + b u = 0 \quad -\infty < x < \infty$$

$$u(x,0) = \varphi(x)$$

Define $v(x,t) = e^{bt} u(x,t)$

Then

$$\partial_t v = b e^{bt} u(x,t) + e^{bt} \partial_t u$$

$$= \cancel{b e^{bt} u(x,t)} + k e^{bt} \partial_x^2 u - \cancel{b e^{bt} u}$$

where the last equality used the PDE for u

But $k e^{bt} \partial_x^2 u = k \partial_x^2 (e^{bt} u) = k \partial_x^2 v$

so

$$\partial_t v = k \partial_x^2 v$$

and

$$v(x,0) = u(x,0) = \varphi(x) \quad (\text{since } e^{bt} = 1 \text{ for } t=0)$$

Thus,

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy$$

where $S(z,t) = \frac{1}{\sqrt{4\pi k t}} e^{-z^2/4kt}$

$\underline{\underline{\text{or}}}$

$$u(x,t) = e^{-bt}\, v(x,t)$$

$$= \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \, \varphi(y)\, dy$$

Solutions for HW#3.

p.58 #3: Since we want the function to have zero derivative at $x=0$, we extend $\varphi(x)$ to the whole line as an even function, i.e.

$$\varphi^e(x) = \begin{cases} \varphi(x) & \text{if } x>0 \\ \varphi(-x) & \text{if } x<0. \end{cases}$$

Then

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi^e(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-(x-y)^2/4kt} \varphi^e(y)\,dy$$

$$+ \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt} \varphi^e(y)\,dy$$

In the second integral $\varphi^e(y) = \varphi(y)$ since $y>0$. In the first integral let $z=-y$. Then $\varphi^e(y) = \varphi^e(-z) = \varphi(z)$.

$$\therefore \quad w(x,t) = \frac{-1}{\sqrt{4\pi kt}} \int_{\infty}^{0} e^{-(x+z)^2/4kt} \varphi(z)\,dz + \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left\{ e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right\} \varphi(y)\,dy$$

Note that $\partial_x w(0,t) = 0$, by explicit computation.

p.87 #1: (a) If one clamps a violin string in the middle that's like imposing Dirichlet b.c.'s at 0 & L/2 instead of 0 & L so the eigenvalues change from

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{to} \quad \tilde{\lambda}_n = \left(\frac{2n\pi}{L}\right)^2$$

This means that the T(t) part of the solution changes from

$$T_n(t) = A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)$$

to

$$\tilde{T}_n(t) = \tilde{A}_n \cos\left(\frac{2n\pi c t}{L}\right) + \tilde{B}_n \sin\left(\frac{2n\pi c t}{L}\right).$$

Note that the frequency of $\tilde{T}_n(t)$ is exactly twice that of $T_n(t)$ & doubling the frequency raises the pitch by an octave.

(b) Increasing the tension increases $c = \sqrt{T/\rho}$ & hence the frequency, since

$$T_n(t) = A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right).$$

p.87 #3

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2} \qquad 0<x<L$$

$$u(0,t) = u(L,t) = 0.$$

Write $u(x,t) = T(t)\,X(t)$. Substituting this into Schrodinger's equation gives

$$\frac{-\dot{T}}{iT} = \frac{-X''}{X} = \lambda.$$

Thus, the X-eqn becomes $X''+\lambda X = 0$; $X(0) = X(L) = 0$ & we know this has non-zero solutions only if $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ & $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

The T equation then becomes

$$\dot{T}_n = -i\lambda_n T_n$$

or

$$T_n(t) = e^{-i\left(\frac{n\pi}{L}\right)^2 t} A_n$$

Thus, we get an ∞-sequence of solutions

$$u_n(x,t) = A_n e^{-i\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

& the general solution will have the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-i\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

p.87 #4 If we set $u(x,t) = T(t)X(x)$ & insert this in the PDE we can rewrite the equation as

$$-\frac{\ddot{T} + r\dot{T}}{c^2 T} = -\frac{X''}{X} = \lambda$$

For the X equation we have (again) solution $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ when $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ Then the T equation becomes

$$\ddot{T}_n + r\dot{T}_n + \left(\frac{n\pi c}{L}\right)^2 T_n = 0$$

If we write $T_n(t) = e^{mt}$ Then m satisfies

$$m^2 + rm + \left(\frac{n\pi c}{L}\right)^2 = 0$$

$$\Rightarrow \quad m = \frac{-r \pm \sqrt{r^2 - 4\left(\frac{n\pi c}{L}\right)^2}}{2}$$

$$= -\frac{1}{2}r \pm \frac{i}{2}\sqrt{\left(\frac{2\pi c n}{L}\right)^2 - r^2}$$

since $\left(\frac{2n\pi c}{L}\right) > r$.

$$\therefore \quad T_n(t) = e^{-rt/2}\left(A_n e^{i\frac{t}{2}\sqrt{\left(\frac{2\pi c n}{L}\right)^2 - r^2}} + B_n e^{-i\frac{t}{2}\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - r^2}}\right)$$

This can also be written in terms of sines & cosines - please accept that form too.

Multiplying by $X_n(x)$ and summing over n we find

$$u(x,t) = \sum_{n=1}^{\infty}\left\{A_n e^{i\frac{t}{2}\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - r^2}} + B_n e^{-i\frac{t}{2}\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - r^2}}\right\} e^{-\frac{tr}{2}} \sin\left(\frac{n\pi x}{L}\right)$$

One then solves for A_n & B_n using the initial conditions as we did for the wave equation.

Page 87 #6: If we assume $u(x,t) = X(x)T(t)$ The equation can be written as

$$X(x)\left(t\frac{dT}{dt}\right) = T(t)\left(\frac{d^2X}{dx^2}\right) + 2T(t)X(x)$$

or
$$\frac{t\frac{dT}{dt} - 2}{T(t)} = \frac{d^2X/dx^2}{X(x)} = -\lambda$$

with $X(0) = X(\pi) = \ell$. The X-part of the problem is the same as we considered in the lecture with $\ell = \pi$ so we have

$$X_n(x) = \sin(nx)$$
$$\lambda_n = n^2 \qquad n = 1, 2, \ldots$$

The t-dependent part of the equation becomes

$$t\frac{dT}{dt} - 2T(t) = -\lambda T(t)$$

or
$$\frac{dT}{dt} = -\frac{(\lambda - 2)}{t} T(t)$$

which has the solution $T(t) = At^{2-\lambda}$. Using the allowed values of λ from above we have

$$T_n(t) = A_n t^{2-n^2}.$$

We're interested in $t = 0$ so note that we can only consider $n = 1$ (since $T_n(0)$ is undefined if $n > 1$.) But for $n = 1$ we have $u_1(x,t) = A_1 t \sin x$ & we get infinitely many solutions (corresponding to different values of A_1) all of which have $u_1(x,0) = 0$.

page 50 #2:

$$u(x,t) = \frac{3}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-(x-y)^2/4kt}\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt}\, dy$$

$$= \frac{3}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-\left(\left(\frac{x}{\sqrt{4kt}}\right) - z\right)^2} dz + \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\left(\left(\frac{x}{\sqrt{4kt}}\right) - z\right)^2} dz$$

$$\underset{w = \left(\frac{x}{\sqrt{4kt}} - z\right)}{=} -\frac{3}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}} e^{-w^2} dw - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-w^2} dw$$

$$= -\frac{3}{\sqrt{\pi}} \left\{ \underbrace{\int_{+\infty}^{0} e^{-w^2} dw}_{= -\frac{\sqrt{\pi}}{2}} + \int_{0}^{\sqrt{4kt}} e^{-w^2} dw \right\}$$

$$- \frac{1}{\sqrt{\pi}} \left\{ \int_{\frac{x}{\sqrt{4kt}}}^{0} e^{-w^2} dw + \underbrace{\int_{0}^{-\infty} e^{-w^2} dw}_{= -\sqrt{\pi}/2} \right\}$$

$$= \frac{3}{2} + \frac{1}{2} - \frac{3}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

$$= 2 - \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$