MA561 Homework #4 Solutions

Page 90. #1

1. If we separate variables as usual we find

$$\frac{\frac{dT}{dt}}{kT(t)} = \frac{\frac{d^2X}{dx^2}}{X} = -\lambda.$$

Solving first for $X(x)$ we have

$$\frac{d^2X}{dx^2} = -\lambda X \; ; \quad X(0) = \frac{dX}{dx}(\ell) = 0$$

As usual we must consider different possibilities for the e'values λ:

<u>Case 1</u>: $\lambda < 0$. Then $\frac{d^2X}{dx^2} = |\lambda| X$ so

$$X(x) = Ae^{\sqrt{|\lambda|}x} + Be^{-\sqrt{|\lambda|}x}.$$

$X(0) = A + B = 0 \Rightarrow B = -A$ so

$$X(x) = A\left(e^{\sqrt{|\lambda|}x} - e^{-\sqrt{|\lambda|}x}\right)$$

Then $X'(\ell) = A\sqrt{|\lambda|}\left(e^{\sqrt{|\lambda|}\ell} + e^{\sqrt{|\lambda|}\ell}\right) = 0$

$\Rightarrow A = 0$ so no negative e'values

Page 90 #1 (cont.)

Case 2: $\lambda = 0$. Then $X(x) = Ax + B$ so

$$X(0) = 0 \Rightarrow \boxed{B = 0}$$

& $X'(\ell) = A = 0 \Rightarrow \boxed{A = 0}$ so $\lambda = 0$ is not an e'value.

Case 3: $\lambda > 0$ In this case $\frac{d^2 X}{dx^2} = -\lambda X$

$\Rightarrow X(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$

$X(0) = 0 = A \Rightarrow \boxed{A = 0}$ so $\boxed{X(x) = B\sin(\sqrt{\lambda}\, x)}$.

$X'(\ell) = B\sqrt{\lambda}\cos(\sqrt{\lambda}\,\ell) = 0.$

If $B \neq 0$ then $\cos(\sqrt{\lambda}\,\ell) = 0 \Rightarrow \sqrt{\lambda}\,\ell = \frac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$

or $$\lambda_n = \frac{(2n-1)^2 \pi^2}{4\ell^2} \qquad n = 1, 2, \ldots$$

$$X_n(x) = \sin\left[\frac{(2n-1)}{2}\frac{\pi x}{\ell}\right] \qquad n = 1, 2, \ldots$$

Now solve the T eqn:

$$\frac{dT_n}{dt} = -\lambda_n k T_n = -\left(\frac{2n-1}{2}\right)^2 \frac{\pi^2}{\ell^2} k\, T(t)$$

or $$T_n(t) = A_n e^{-k\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{\ell}\right]^2 t}.$$

Page 90 #1 (cont.) Thus,

$$u_n(x,t) = A_n e^{-k\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{\ell}\right]^2 t} \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{\ell}\right]$$

& the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{\ell}\right]^2 t} \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{\ell}\right].$$

Page 90 #2: If we separate variables as above we find

$$\frac{\frac{d^2T}{dt^2}}{c^2T(t)} = \frac{\frac{d^2X}{dx^2}}{X(x)} = -\lambda$$

The eigenvalue problem is almost the same as in the previous problem except that the roles of $x=0$ & $x=\ell$ are reversed. As above we find that

$$\lambda_n = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{\ell}\right]^2 \quad n=1,2,\ldots$$

&

$$X_n(x) = \cos\left[\left(\frac{2n-1}{2}\right)\frac{\pi x}{\ell}\right].$$

The T-eqn becomes

$$\frac{d^2T_n}{dt^2} = \left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi c}{\ell}\right)\right] T_n$$

whose solution is:

page 90 #2 (cont.)

$$T_n(t) = A_n \cos\left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi c t}{\ell}\right)\right] + B_n \sin\left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi c t}{\ell}\right)\right]$$

Thus, the general solution in this case is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{A_n \cos\left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi c t}{\ell}\right)\right] + B_n \sin\left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi c t}{\ell}\right)\right] \times \right.$$

$$\times \cos\left[\left(\frac{2n-1}{2}\right)\left(\frac{\pi x}{\ell}\right)\right].$$

p.90#3: $u_t = k\,u_{xx}$ $\quad -L \le x \le L$

$$u(-L,t) = u(L,t)\ ;\ u_x(-L,t) = u_x(L,t).$$

Separating variables as usual one finds

$$-\frac{\dot{T}}{kT} = \frac{-X''}{X} = \lambda$$

Thus, the e'value equation becomes

$$X'' + \lambda X = 0\ ;\ X(-L) = X(L)\ ;\ X'(-L) = X'(L)$$

As usual, one can show there are no solutions with $\lambda < 0$.

Case 1: $\lambda = 0$. Then $X''(x) = 0 \Rightarrow X(x) = A + Bx$.
Applying the b.c.'s we find $\lambda_0 = 0$ is an e'value with e'function $X_0(x) = \text{const}$

Case 2 $\lambda > 0$. In this case $X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$
The b.c.'s imply

$$A\cancel{\cos(\sqrt{\lambda}L)} - B\sin(\sqrt{\lambda}L) = A\cancel{\cos(\sqrt{\lambda}L)} + B\sin(\sqrt{\lambda}L)$$
$$A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + \cancel{B\sqrt{\lambda}\cos(\sqrt{\lambda}L)} = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) + \cancel{B\sqrt{\lambda}\cos(\sqrt{\lambda}L)}$$

or

$$2B\sin(\sqrt{\lambda}L) = 0$$
$$2A\sin(\sqrt{\lambda}L) = 0$$

$\therefore$ If $\sqrt{\lambda_n}\,L = n\pi$, or $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$

both A & B can be non-zero & hence the e'function is

$$\underline{X}_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right).$$

Now inserting the values of λ_n into the T eqn we find

$$T_n(t) = \begin{cases} \text{const} & \text{if } n=0 \\ C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} & n=1,2,\dots \end{cases}$$

Thus, we get an infinite sequence of solutions

$$u_n(x,t) = \begin{cases} \frac{1}{2}A_0 \text{ (or any other const.)} & \text{if } n=0 \\ \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} & \text{if } n=1,2,\dots \end{cases}$$

Taking the sum of the u_n's gives the form of the solution stated in the text.

<u>page 97 #1:</u> $\quad \underline{X}'' + \lambda \underline{X} = 0 \; ; \; \underline{X}(0)=0, \; \underline{X}'(L) + a\underline{X}(L) = 0.$

<u>Case 1</u>: $\lambda < 0$. Then $\underline{X}(x) = A\cosh(\sqrt{|\lambda|}\,x) + B\sinh(\sqrt{|\lambda|}\,x)$

The b.c.'s imply

$$\underline{X}(0) = A = 0$$

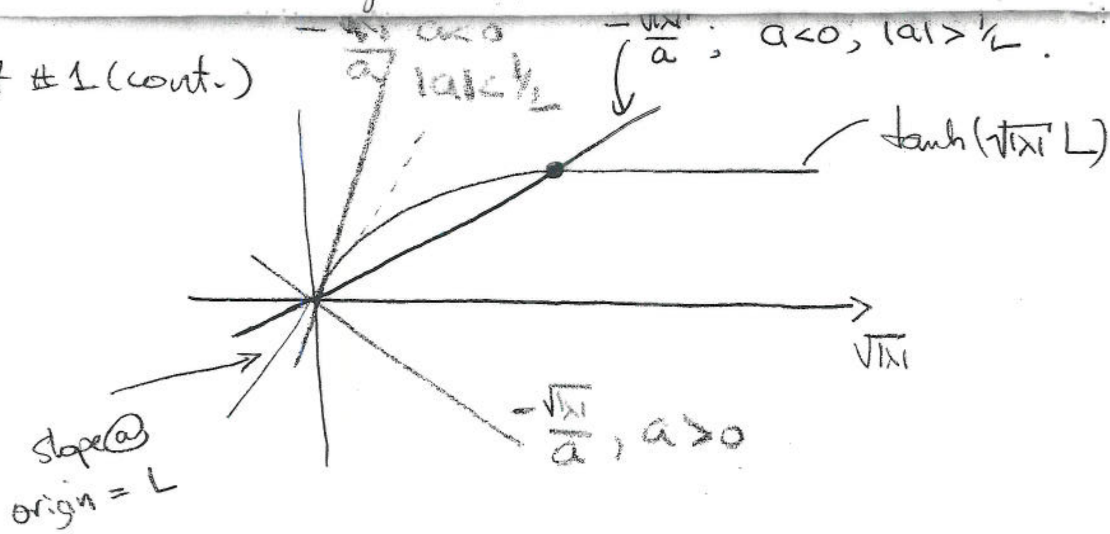
~~which~~ so $\underline{X}(x) = B\sinh(\sqrt{|\lambda|}\,x)$. The b.c. @ $x=L$ then gives

$$B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\,L) + aB\sinh(\sqrt{|\lambda|}\,L) = 0$$

We get a non-zero solution B iff.

$$-\frac{\sqrt{|\lambda|}}{a} = \tanh(\sqrt{|\lambda|}\,L)$$

p.97 #1 (cont.)



From the graph we see that there is one negative eigenvalue if $a < -1/L$, & no neg. e'values if $-1/L < a < 0$ or $a > 0$.

Case 2: $\lambda = 0$ Then $X(x) = A + Bx$.

Since $X(0) = 0$, $A = 0$. At $x = L$ we have

$$B + aBL = 0$$

This has a non-zero solution B only if $1 + aL = 0$ or

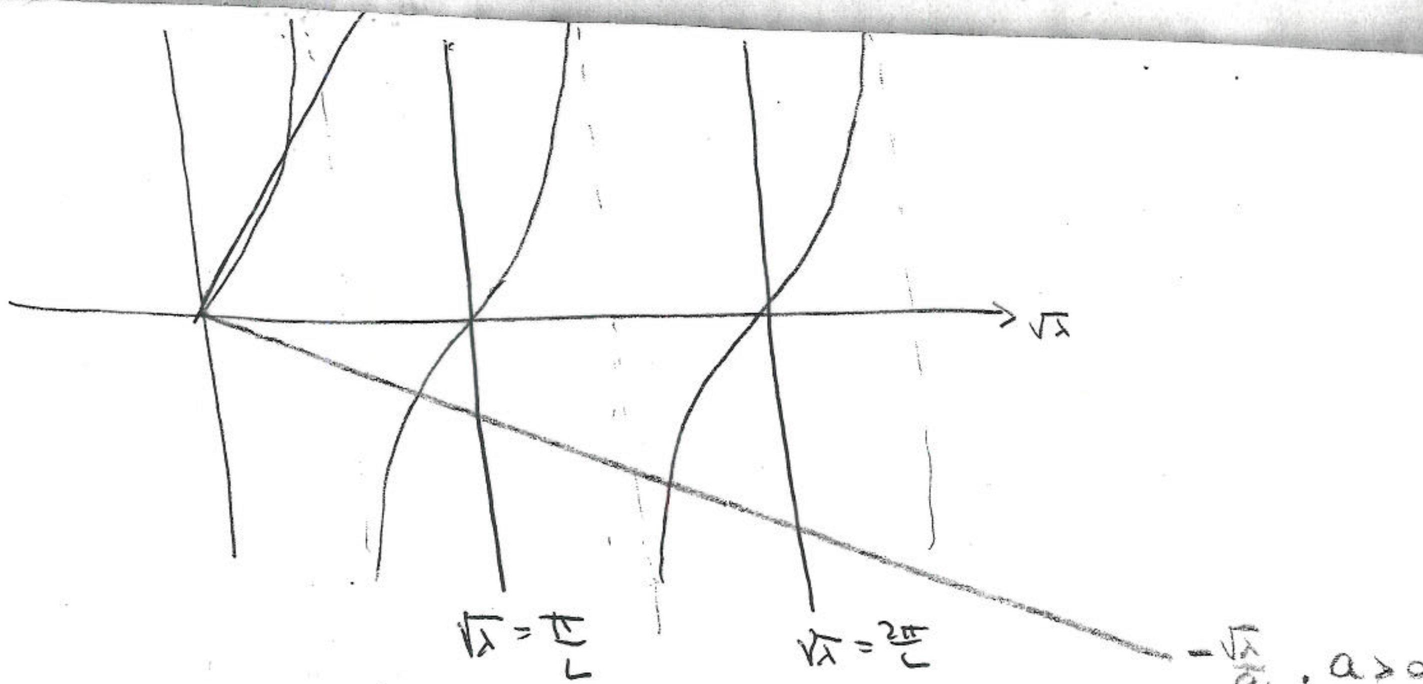
$$\boxed{a = -1/L}$$

Case 3: $\lambda > 0$. In this case $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$
$X(0) = 0 \Rightarrow A = 0$. The b.c. at $x = L$ implies

$$\sqrt{\lambda}\,B\cos(\sqrt{\lambda}L) + aB\sin(\sqrt{\lambda}L) = 0$$

& hence there is a non-zero solution B only if

$$-\frac{\sqrt{\lambda}}{a} = \tan(\sqrt{\lambda}L)$$



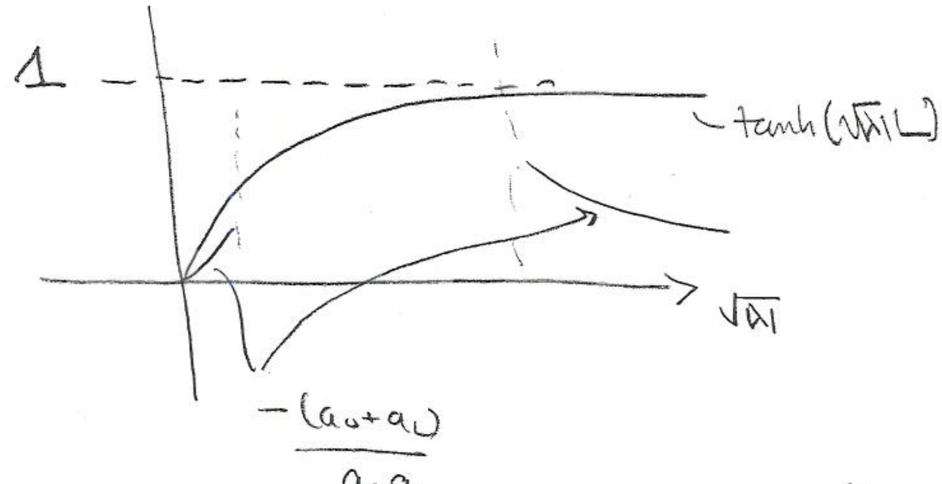
If $a > 0$ or $-\frac{1}{L} < a < 0$, there is a positive eigenvalue $\sqrt{\lambda_n} \in \left(\frac{n\pi}{L}, \frac{(n+1)\pi}{L}\right)$, $n = 0, 1, 2 \ldots$
If $a < -1/L$, then there is no eigenvalue in the interval $(0, \pi/L)$, but we do have e'values satisfying $\sqrt{\lambda_n} \in \left(\frac{n\pi}{L}, \frac{(n+1)\pi}{L}\right)$ for $n = 1, 2, \ldots$

page 98, # 4: Proceeding as in the lecture we deduce that any negative eigenvalue must satisfy the equation

$$-\frac{(a_0 + a_L)\sqrt{|\lambda|}}{|\lambda| + a_0 a_L} = \tanh(\sqrt{|\lambda|}\, L).$$

The slope of the RHS of this equation at the origin is L. The slope of the RHS of this equations at the origin is $-\frac{(a_0 + a_L)}{a_0 a_L} < L$ by the hypotheses of the problem.

p.98 #4 (cont.) Thus, near the origin, the graph of the left and right hand sides of the equation look like



We now determine the behavior of $-\frac{(a_0+a_L)\sqrt{|\lambda|}}{|\lambda|+a_0a_L}$ for the intermediate region. We look at the max of

$$f(\beta) = \frac{-(a_0+a_L)\beta}{\beta^2+a_0a_L} \qquad (\text{where } \beta = \sqrt{|\lambda|})$$

$$f'(\beta) = \frac{-(a_0+a_L)}{\beta^2+a_0a_L} + \frac{2(a_0+a_L)\beta^2}{(\beta^2+a_0a_L)^2} = 0$$

$$\Leftrightarrow \quad -(a_0+a_L)\beta^2 - a_0a_L(a_0+a_L) + 2(a_0+a_L)\beta^2 = 0$$

$$\underline{\text{or}} \qquad \boxed{\beta^2 = a_0a_L}$$

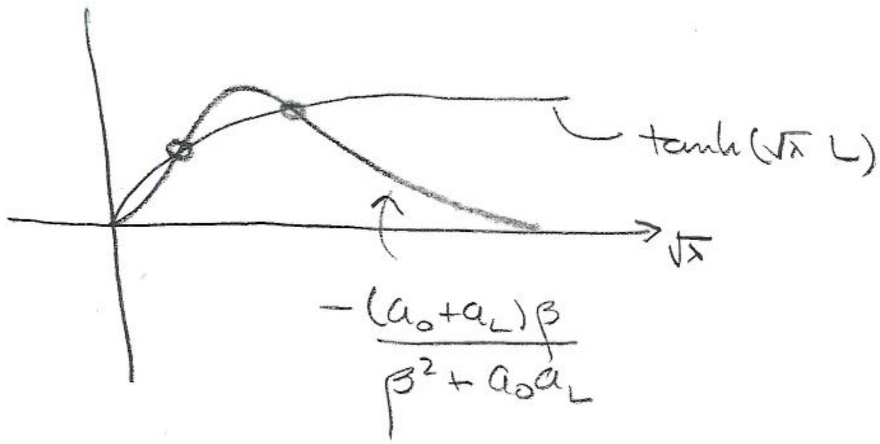
At this critical point

$$f(\sqrt{a_0a_L}) = \frac{-(a_0+a_L)\sqrt{a_0a_L}}{2a_0a_L} = \frac{-(a_0+a_L)}{2\sqrt{a_0a_L}}$$

Note that if $c = \sqrt{-a_0}$, $d = \sqrt{-a_L}$, this last expression is of the form

$$\frac{c^2 + d^2}{2cd} \geq 1$$

Thus, the max of $f(\sqrt{\lambda})$ is greater than or equal to one and hence at this point it will be strictly greater than $\tanh(\sqrt{|\lambda|} L)$. Thus, by the intermediate value theorem the graphs of $f(\sqrt{\lambda})$ and $\tanh(\sqrt{\lambda} L)$ must intersect at least twice. (In fact they intersect exactly twice since $f(\sqrt{\lambda})$ is monotonic increasing from 0 to its max, and then monotonic decreasing.)



p. 108 #2: From the Maple worksheet below we see that the coefficients of the Fourier sine series of $\varphi(x) = x^2$ $(0 \le x \le 1)$ are

$$B(n) = \frac{(-2)}{n^3\pi^3}\left(2 - 2(-1)^n + (n^2\pi^2)(-1)^n\right)$$

$\therefore$ The Fourier-sine series is

$$\sum_{n=1}^{\infty} \frac{(-2)}{n^3\pi^3}\left(2 - (-2)^n + (-1)^n(n^2\pi^2)\right) \sin(n\pi x)$$

Similarly the Fourier cosine coefficients are

$$A(n) = 4\left(\frac{(-1)^n}{n^2\pi^2}\right)$$

so the Fourier cosine is

$$\frac{1}{2}\cdot\frac{2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2}\cos(n\pi x)$$

This is $A_0 = \frac{2}{1}\int_0^1 x^2\,dx = \frac{2}{3}$

```
> f(x):=x^2;
```

$$f(x) := x^2$$

```
> B(n):=2*int(f(x)*sin(n*Pi*x),x=0..1);
```

$$B(n) := -2\,\frac{n^2\pi^2\cos(n\pi) - 2\cos(n\pi) - 2\,n\,\pi\sin(n\pi) + 2}{n^3\pi^3}$$

```
> A(n):=2*int(f(x)*cos(n*Pi*x),x=0..1);
```

$$A(n) := 2\,\frac{n^2\pi^2\sin(n\pi) - 2\sin(n\pi) + 2\,n\,\pi\cos(n\pi)}{n^3\pi^3}$$

p.108 #3: This is solved with Maple below. I have chosen $L=1$ for simplicity. By problem 7 on p.113 we know we can always convert a Fourier series on $(0,L)$ to $(0,1)$ just by changing the scale so this involves no loss of generality.

```
> g(x):=x;
```

$$g(x) := x$$

```
> B(n):=2*int(g(x)*sin(n*Pi*x),x=0..1);
```

$$B(n) := -2\,\frac{-\sin(n\pi) + n\pi\cos(n\pi)}{n^2\pi^2}$$

```
> A(n):=2*int(g(x)*cos(n*Pi*x),x=0..1);
```

$$A(n) := 2\,\frac{\cos(n\pi) + n\pi\sin(n\pi) - 1}{n^2\pi^2}$$

```
> A0:=2*int(g(x),x=0..1);
```

$$A0 := 1$$

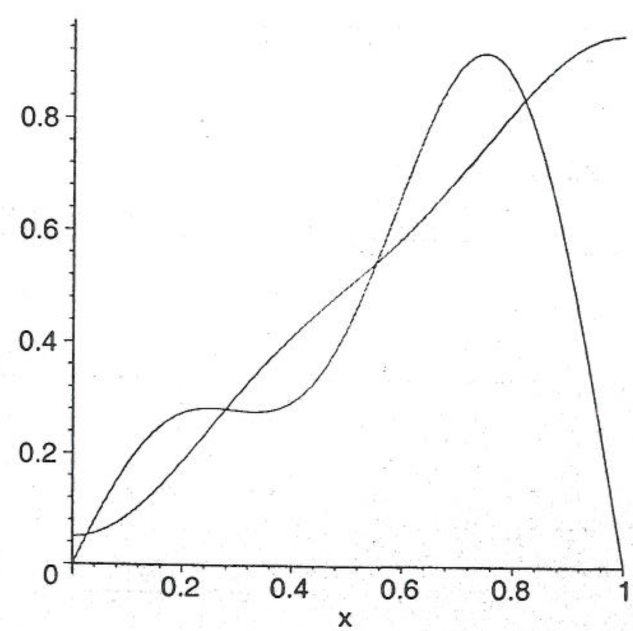
```
> S3(x):=sum(B(n)*sin(n*Pi*x),n=1..3);
```

$$S3(x) := 2\,\frac{\sin(\pi x)}{\pi} - \frac{\sin(2\pi x)}{\pi} + \frac{\frac{2}{3}\sin(3\pi x)}{\pi}$$

```
> C3(x):=(A0/2)+sum(A(n)*cos(n*Pi*x),n=1..3);
>
```

$$C3(x) := \frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4}{9}\,\frac{\cos(3\pi x)}{\pi^2}$$

```
> g1:=plot(S3(x),x=0..1):
> g2:=plot(C3(x),x=0..1):
> display({g1,g2});
```

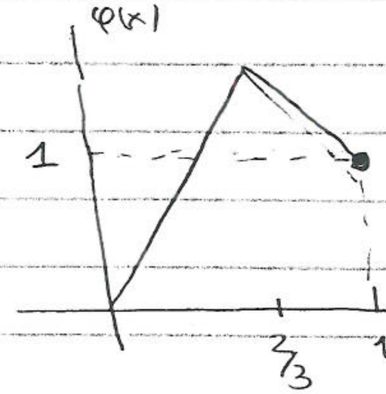


p.108 #8:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1$$

$u(0,t) = 0,\ u(1,t) = 1$

$u(x,0) = \varphi(x),$



where $\varphi(x) = \begin{cases} 5x/2 & 0 < x < 2/3 \\ 3 - 2x & \frac{2}{3} < x < 1 \end{cases}$

We can't apply the method of separation of variables directly since we don't have zero Dirichlet b.c.'s. However, The $U(x) = x$. Note that $U(x)$

(i) Solves the heat equation (it's the steady state solution)

(ii) Satisfies the b.c.'s.

Define $w(x,t) = u(x,t) - U(x)$. Then w still solves the heat equation (by linearity); i.e

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \qquad 0 < x < 1$$

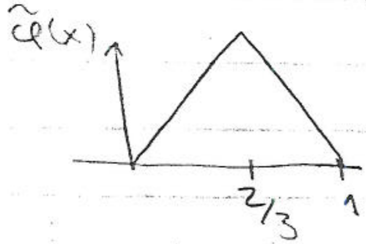
But $w(0,t) = u(0,t) - U(0) = 0$

$w(1,t) = u(1,t) - U(1) = 1 - 1 = 0$

i.e. w satisfies zero Dirichlet b.c.'s.

The initial condition for w is

$$w(x,0) = u(x,0) - U(x) = \tilde{\varphi}(x) = \begin{cases} \frac{3x}{2} & 0 < x < 2/3 \\ 3 - 3x & 2/3 < x < 1 \end{cases}$$



We can solve for $W(x,t)$ using the method of separation of variables & express the solution as

$$W(x,t) = \sum_{n=1}^{\infty} B_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

To find the B_n's we set $t=0$ so

$$W(x,0) = \tilde{\varphi}(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

$$\Rightarrow B_n = 2\int_0^1 \tilde{\varphi}(x) \sin(n\pi x)\,dx = \frac{9 \cdot \sin(2n\pi/3)}{n^2\pi^2}$$

(using Maple)

Now we can express our original function $u(x,t)$ as

$$u(x,t) = W(x,t) + U(x) = x + \sum_{n=1}^{\infty} \frac{9 \cdot \sin\left(\frac{2\pi n}{3}\right)}{n^2\pi^2} e^{-(n\pi)^2 t} \sin(n\pi$$

p.108

page 108 #9. $u_{tt} = c^2 u_{xx} \quad 0<x<\pi$

$u_x(0,t) = u_x(\pi,t) = 0.$

Let $u(x,t) = X(x)\,T(t)$. Then

$$X(x)\,\ddot{T}(t) = c^2\, T(t)\, X''(x)$$

or

$$-\frac{\ddot{T}(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

The X equation becomes

$$X'' + \lambda X = 0 \qquad 0<x<\pi$$

$$X'(0) = X'(\pi) = 0$$

We solved this problem in lecture and we know that the e'values are

$$\lambda_n = n^2,\ n = 0,1,2,\ldots$$

with e'functions

$$X_n(x) = \begin{cases} 1/2 & \text{if } n=0 \\ \cos(nx) & \text{if } n = 1,2,\ldots \end{cases}$$

Now solve the T equation

$$\ddot{T}_n + (nc)^2\, T_n = 0$$

$$\Rightarrow \quad T_n(t) = \begin{cases} A_n \cos(nt) + B_n \sin(nt) & (n = 1,2\ldots) \\ A_0 + B_0 t & (n=0). \end{cases}$$

p. 108 #9 (cont.)

Thus, we get a sequence of solutions

$$u_n(x,t) = \begin{cases} \frac{1}{2}(A_0 + B_0 t) & n=0 \\ \left(A_n \cos(cnt) + B_n \sin(cnt)\right) \cos(nx) & \end{cases}$$

Taking a sum of these separated solutions we find the general form of the solution is

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \left(A_n \cos(cnt) + B_n \sin(cnt)\right) \cos(nx).$$

Now apply the initial conditions to solve for A_n, B_n:

$$u(x,0) = 0 = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

$$\Rightarrow \boxed{A_n = 0 \quad \forall n}$$

Now set

$$\frac{\partial u}{\partial t}(x,0) = \boxed{\cos^2 x = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} cnB_n \cos(nx)}$$

If we write

$$\cos^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

we see that $B_0 = 1$, $2cB_2 = -\frac{1}{2} \Rightarrow B_2 = -\frac{1}{4c}$ &
all other B_n's are zero. Thus

$$\boxed{u(x,t) = \frac{1}{2}t - \frac{1}{4c}\cos(2ct)\cos(2x)}$$

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[redacted]

p. 113 #1

(a) $\sin(ax)$; odd - periodic with period $2\pi/a$

(b) e^{ax} ; neither odd nor even ; not periodic

(c) x^m ; odd if m is odd, even if m is even ; not periodic

(d) $\tan x^2$; even but not periodic

(e) $|\sin(x/6)|$ even with period $\pi/6$

(f) $x\cos(ax)$. odd but not periodic.

#7. Suppose $\tilde{f}(x')$ is defined for $-\pi < x' < \pi$, with Fourier series

$$(**)' \qquad \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(nx') + B_n \sin(nx') \right)$$

Setting $x' = \frac{\pi x}{L}$ we have $f(x) \equiv \tilde{f}\left(\frac{\pi x}{L}\right)$ defined for $-L < x < L$ & the Fourier series $(**)'$ become

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

which is a Fourier series on $(-L, L)$. The

p. 113 #7 (cont.) last thing to check is that A_n & B_n are the Fourier coefficients for f. We have

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x') \sin(nx')\,dx'$$

$$\underset{x' = \frac{\pi x}{L}}{=} \frac{1}{L}\int_{-L}^{L} \tilde{f}\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

– i.e. the B_n's are the Fourier sine coefficients of $f(x)$. The proof for the coefficients A_n is almost identical.