p.114 #.11 The complex form of Fourier series of $f(x)=e^x$ on $(-L,L)$ has coefficients

Solutions for HW #5

$$C_n = \frac{1}{2L}\int_{-L}^{L} e^{x} e^{-i\left(\frac{n\pi}{L}\right)x}\,dx$$

$$= \frac{1}{2L}\int_{-L}^{L} e^{\left(1-i\left(\frac{n\pi}{L}\right)\right)x}\,dx$$

$$= \frac{1}{2L}\frac{1}{\left(1-i\left(\frac{n\pi}{L}\right)\right)} e^{\left(1-i\left(\frac{n\pi}{L}\right)\right)x}\Bigg|_{-l}^{L}$$

$$= \frac{1}{2L}\frac{1}{\left(1-i\left(\frac{n\pi}{L}\right)\right)}\left(e^{L}e^{-in\pi} - e^{-L}e^{in\pi}\right)$$

But $e^{\pm in\pi} = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd}\end{cases} = (-1)^n$

It's OK to leave this in exponential form & not introduce sinh.

$$= \frac{1}{2L}\frac{(-1)^n}{1-i\left(\frac{n\pi}{L}\right)}\left(e^{L}-e^{-L}\right) = \frac{1}{L}\frac{(-1)^n\sinh(L)}{\left(1-i\left(\frac{n\pi}{L}\right)\right)}$$

$$\therefore\quad e^x \sim \sum_{n=-\infty}^{\infty} \frac{1}{L}\frac{(-1)^n\sinh(L)}{\left(1-i\left(\frac{n\pi}{L}\right)\right)} e^{i\left(\frac{n\pi}{L}\right)x} \quad (*)$$

To derive the real form of the Fourier series we can either evaluate the integrals over $\int_{-L}^{L} e^x\cos\left(\frac{n\pi x}{L}\right)dx$ & similarly for the sine function or we can rewrite $(*)$ as

$$\frac{\sinh(L)}{L} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{L}\sinh(L)\left\{\frac{e^{i\left(\frac{n\pi}{L}\right)x}}{1-i\left(\frac{n\pi}{L}\right)} + \frac{e^{-i\left(\frac{n\pi}{L}\right)x}}{1+i\left(\frac{n\pi}{L}\right)}\right\}\right)$$

p.114 #11 (cont.)

Combining the expressions in {..} we have

$$\frac{(1+i(\frac{n\pi}{L}))e^{i(\frac{n\pi}{L})x} + (1-i(\frac{n\pi}{L}))e^{-i(\frac{n\pi}{L})x}}{1+(\frac{n\pi}{L})^2}$$

$$= \frac{\left\{\cos(\frac{n\pi x}{L}) - (\frac{n\pi}{L})\sin(\frac{n\pi x}{L}) + \cancel{i(\sin(\frac{n\pi x}{L}))} + i\cancel{(\frac{n\pi}{L})\cos(\frac{n\pi x}{L})} + \cos(\frac{n\pi x}{L}) - (\frac{n\pi}{L})\sin(\frac{n\pi x}{L}) - \cancel{i\sin(\frac{n\pi x}{L})} - i\cancel{(\frac{n\pi}{L})\cos(\frac{n\pi x}{L})}\right\}}{1+(\frac{n\pi}{L})^2}$$

$$= \frac{2}{1+(\frac{n\pi}{L})^2}\cos\left(\frac{n\pi x}{L}\right) - \frac{2(\frac{n\pi}{L})}{1+(\frac{n\pi}{L})^2}\sin\left(\frac{n\pi x}{L}\right)$$

Thus,

$$e^x \sim \frac{\sinh(L)}{L} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n 2\sinh(L)}{L(1+(\frac{n\pi}{L})^2)}\cos\left(\frac{n\pi x}{L}\right) - \frac{(-1)^n 2\sinh(L)}{L(1+(\frac{n\pi}{L})^2)}\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right\}$$

Which is the trigonometric form of F.S. of e^x.

p. 119 #6 $\quad \frac{dX}{dx} = \lambda X$

The equation has solution $X(x) = ce^{\lambda x}$ for any complex number λ. If we impose the boundary condition

$$X(0) = X(1) \Rightarrow c = ce^{\lambda} \quad \text{or} \quad \boxed{e^{\lambda} = 1}$$

From this equation we see that the eigenvalues are

$$\boxed{\lambda_n = 2\pi i n \quad ; \; n = 0, \pm 1, \pm 2, \ldots}$$

To check the orthogonality compute

$$\int_0^1 X_n(x)\overline{X_m(x)}\,dx = \int_0^1 e^{2\pi i n x}\, e^{-2\pi i m x}\,dx$$

$$= \int_0^1 e^{2\pi i (n-m)x}\,dx = \frac{1}{2\pi i(n-m)} e^{2\pi i (n-m)x}\Big|_0^1$$

assuming $m \neq n$

$$= \frac{1}{2\pi i(n-m)}\left(e^{2\pi i (n-m)} - 1\right) = 0 \text{ if } n \neq m$$

since $e^{2\pi i l} = 1$ for any integer l.

Prob #1. Recall from calculus that the partial sums of the series are

$$S_N(x) = \sum_{n=0}^{N} (-x^2)^n = \frac{1-(-x^2)^N}{1+x^2}$$

For $|x| < 1$, $(-x^2)^N \to 0$ as $N \to \infty$. Thus,

(a) $$S_N(x) \xrightarrow[\text{pointwise}]{} \frac{1}{1+x^2} \quad \text{for all } x \in (-1,1)$$

(b) $S_N(x)$ does not converge uniformly to $\frac{1}{1+x^2}$ on the interval $(-1,1)$. If it did, for every $\epsilon > 0$ we would be able to Find some $N_\epsilon > 0$ s.t. for all $N > N_\epsilon$,

$$\left| \frac{1}{1+x^2} - S_N(x) \right| < \epsilon. \quad \text{for } \underline{\text{all}} \ x \in (-1,1).$$

But if we pick $\epsilon < \frac{1}{4}$ this cannot be satisfied since if N is even, we can make $1/1+x^2$ arbitraily close to $1/2$ & $S_N(x)$ arbitraily close to zero by picking x close to one and hence their difference can't be less than ϵ.

p. 129 #1 (cont)

(c)

$$\int_{-1}^{1}\left(S_N(x)-\frac{1}{1+x^2}\right)^2 dx = \int_{-1}^{1}\left(\frac{(-x^2)^N}{1+x^2}\right)^2 dx$$

$$= \int_{-1}^{1}\frac{x^{4N}}{1+x^2}\,dx \leq \int_{-1}^{1} x^{4N}\,dx$$

$$= \frac{1}{4N+1}x^{4N}\Big|_{-1}^{1} = 2/4N+1$$

Thus, $\lim_{N\to\infty}\int_{-1}^{1}\left(S_N(x)-\frac{1}{1+x^2}\right)^2 dx = 0$

so the geometric series does converge in the L^2 sense on $(-1,1)$.

p. 130 #5: (a) (see Maple worksheet)

(b) For $0<x<1$ and $1<x<3$ the function is cont. & hence the F.S. converges to $\varphi(x)$ i.e. The limit of the series is 0 if $0<x<1$ and the limit of the series is 1 if $1<x<3$.

At $x=1$ the series converges to $\frac{1}{2}(\varphi(1^+)+\varphi(1^-)) = 1/2$.

At the endpoints of the interval (i.e. $x=0$ or $x=3$) the F.S. will converge to the even periodic extension of $\varphi(x)$ which is continuous so at 0, the Fourier series converges to zero and at $x=3$ it converges to 1.

p.130 #5 (c) Yes - Since φ is p.w. continuous, &

$$\int_0^3 (\varphi(x))^2\,dx = 2 < \infty$$ we know that the

F.S. converges in the L^2 sense.

(d) Note that when $x=0$, the Fourier cosine series is

$$\frac{2}{3} - \frac{\sqrt{3}}{\pi} - \frac{\sqrt{3}}{\pi}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{\pi}\left(\frac{1}{4}\right) + \frac{\sqrt{3}}{\pi}\left(\frac{1}{5}\right) - \frac{\sqrt{3}}{\pi}\left(\frac{1}{7}\right) - \frac{\sqrt{3}}{\pi}\left(\frac{1}{8}\right) + \cdots$$

Recalling from part (b) that this converges to zero we have

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \cdots = \frac{2}{3}\cdot\frac{\pi}{\sqrt{3}}$$

p.130 #8 (a) Since $f(x)$ and $f'(x)$ is cont. for $0<x<\ell$ The Fourier series will converge in the pointwise sense for all $x \in (0,\ell)$. The Fourier series will not converge uniformly because the odd periodic extension of the function is discontinuous at $\pm L$. The Fourier series will converge in the L^2 sense since $\int_0^L x^6\,dx = \frac{1}{7}L^7 < \infty$.

(b) Note that $f(x)$ $f'(x)$ are cont. for $0<x<\ell$ & hence the ~~sense~~ Fourier sine series will converge in the pointwise sense for all $x \in (0,\ell)$.
Furthermore by Theorem 2 of this section the convergence

will be uniform. Also, $\int_0^L (Lx - x^2)^2 dx < \infty$ so the series converges in L^2.

ⓒ The coefficients in the Fourier sine series for $f(x) = x^{-2}$ would be

$$B_n = \frac{2}{L} \int_0^L \frac{\sin\left(\frac{n\pi x}{L}\right)}{x^2} dx$$

However, this integral is divergent so the Fourier series does not exist & hence cannot converge in any sense.

in the Fourier sine series are

$$B_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left\{ -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \cancelto{0}{\frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx} \right\}$$

int. by parts

$$= \frac{2L}{n\pi}(-1)^{n+1}$$

Thus, by Parseval's equality

$$\sum_{n=1}^{\infty} \frac{4L^2}{\pi^2 n^2} = \frac{2}{L}\int_0^L x^2 dx = \frac{2}{3L}\cdot L^3 = \frac{2L^2}{3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6}$$