p. 154 #5:

If we write the Laplacian in polar coordinates we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1$$

Because of the symmetry of the situation we expect the solution of the equation to be independent of θ.
i.e.

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial u}{\partial r}\right) = 1.$$

Note that this is a first order linear ordinary differential equation for $\partial u/\partial r$ so we can immediately write down its solution:

$$\frac{\partial u}{\partial r} = \frac{c_1}{r} + \frac{1}{2}r$$

Integrating this again gives

$$u(r) = c_1 \ln r + \frac{r^2}{4} + c_2$$

In order for this solution to be continuous at the origin we must choose $c_1 = 0$.

Thus $u(r) = \frac{1}{4}r^2 + c_2$. In order to satisfy the boundary condition when $r = a$, namely

$u(a) = \frac{1}{4}a^2 + c_2 = 0$ we take $c_2 = -a^2/4$ and hence have

$$u(r) = \frac{1}{4}(r^2 - a^2)$$

$$u(x,y) = \frac{1}{2}(x^2 - y^2) - ax + by + a_{00}$$

(The solution of Laplace's equation with Neumann b.c.'s is not unique - that's why we can add an arbitrary constant.)

p.154 Problem 11:

From the equation we have

$$\iiint_D f \, dx\,dy\,dz = \iiint_D \Delta u \, dx\,dy\,dz$$

$$= \iiint_D \operatorname{div}(\nabla u) \, dx\,dy\,dz$$

$$\overset{\text{divergence theorem}}{=} \iint_{\partial D} (\nabla u \cdot \hat{n}) \, ds$$

But $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial n} = g$. Thus,

$$\iiint_D f \, dx\,dy\,dz = \iint_{\partial D} g \, ds,$$

if a solution exists.

...II: Solve $u_{xx}+u_{yy}=0$ on the rectangle $0<x<a$ and $0<y<b$ with the following boundary conditions:

$$u_x = -a \quad \text{on } x=0 \qquad\qquad u_y = b \quad \text{on } y=0$$
$$u_x = 0 \quad \text{on } x=a \qquad\qquad u_y = 0 \quad \text{on } y=b$$

If we use the hint that the solution should be a quadratic polynomial in x & y we can look for a solution of the form

$$u(x,y) = a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}$$

In order to solve Laplace's equation we must have

$$u_{xx}+u_{yy} = 2a_{20}+2a_{02}=0 \Rightarrow \boxed{a_{20} = -a_{02}}$$

Thus, $u(x,y) = a_{20}(x^2-y^2) + a_{11}xy + a_{10}x + a_{01}y + a_{00}$

ext, since $u_x(a,y) = 2a_{20}\,a + a_{11}y + a_{10} = 0$ for all $0<y<b$
we see that $a_{11}=0$ and $a_{10} = -2a_{20}\,a$

Thus. $u(x,y) = a_{20}(x^2-y^2) - 2a\,a_{20}x + a_{01}y + a_{00}$

ext

$$u_y(x,b) = -2a_{20}b + a_{01} = 0 \Rightarrow a_{01} = 2b\,a_{20}$$

$$u(x,y) = a_{20}(x^2-y^2) - 2a\,a_{20}x + 2b\,a_{20}y + a_{00}$$

ext apply $u_x(0,y) = -2a\,a_{20} = -a \Rightarrow a_{20} = \frac{1}{2}$

$u_y(x,0) = 2b\,a_{20} = b \Rightarrow a_{20} = \frac{1}{2}$.

p. 158 #3. Using the method of separation of variables as we did in the lecture one can write the solution

$$u(x,y) = \frac{1}{2}B_0 x + \sum_{n=1}^{\infty} B_n \sinh(nx)\cos(ny)$$

which satisfies Laplace's equation and the three homogeneous boundary conditions. Applying the boundary condition at $x=\pi$ we see

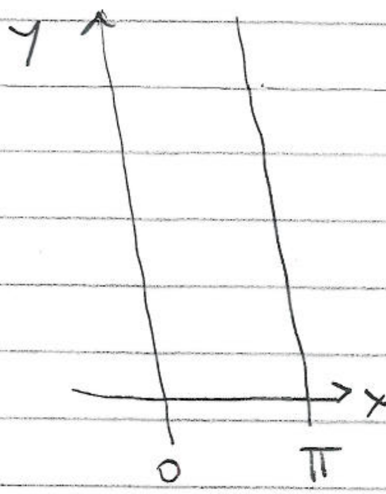
$$u(\pi,y) = \frac{1}{2}B_0\pi + \sum_{n=1}^{\infty} B_n \sinh(n\pi)\cos(ny) = \frac{1}{2} + \frac{1}{2}\cos(2y)$$

so $B_0 = \frac{1}{\pi}$, $B_2 = \frac{1}{2\sinh(n\pi)}$ and all other B_n's are zero.

$$\boxed{u(x,y) = \frac{x}{2\pi} + \frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}}$$

Page 59 #7:

Applying the method of separation of variables we find



$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda.$$

As usual we find the solution of the X eqn to be

$$X_n(x) = \sin(nx)$$
$$\lambda_n = n^2$$

Thus,

$$Y_n''(y) = n^2 Y_n(y)$$

$$\Rightarrow Y_n(y) = A_n e^{ny} + B_n e^{-ny},$$

Since We want $u_n(x,y) \to 0$ as $y \to \infty$, $A_n = 0$

$\therefore$ $u_n(x,y) = B_n e^{-ny} \sin(nx).$

The general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-ny} \sin(nx)$$

To solve for B_n use the condition at $y=0$

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$$u(x,0) = h(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

$$\Rightarrow B_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)\,dx.$$

p.163 #1.

(a) By the max. principle the max. must occur on the boundary. On the boundary $u(2,\theta) = 3\sin(2\theta)+1$ whose max is 4.

$\therefore$ max of u on $\bar{D} = 4$.

(b) By the MVP the value @ the origin equals the average around the boundary so

$$u(0,0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} (1+3\sin(2\theta))\,d\theta = 1.$$

p.164 #2. From formula (10) on p. 160

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

When $r=a$ we have

$$u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta))$$

$$= 1+3\sin\theta \Rightarrow A_0 = 2 \text{ all other } A_n\text{'s} = 0$$

$$B_1 = 3/a \text{ all other } B_n\text{'s} = 0$$

$\therefore$ $u(r,\theta) = 1 + 3\frac{r}{a}\sin(\theta)$.

p. 168 #5: (a) If we apply the method of separation of variables the equations for R & Θ are the same as in a disc and in particular we find

$$\Theta_n(\theta) = \begin{cases} 1 & \text{if } n=0 \\ A_n\cos(n\theta)+B_n\sin(n\theta) & n=1,2,\dots \end{cases}$$

with $\lambda_n = n^2$. The R equation then becomes

$$r^2R_n'' + rR_n' = n^2R_n$$

$n=0$: In this case

$$R_0(r) = C_0 + D_0 \ln r.$$

The b.c. at $r=2$ is $R'(2)=0$ (insulating, or Neuman b.c.) which implies $D_0=0$ so

$$R_0(r) = C_0.$$

$n=1,2,\dots$ Then $R_n(r) = C_nr^n + D_nr^{-n}$

In this case the b.c. @ $r=2$ is

$$R_n'(2) = C_n \cdot n \cdot 2^{n-1} - nD_n2^{-n-1} = 0$$

$$\Rightarrow D_n = 2^{2n}C_n$$

so $R_n(r) = C_n(r^n + 2^{2n}r^{-n})$.

Thus, we obtain a general solution of the form

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} C_n(r^n + 2^{2n}r^{-n})(A_n\cos(n\theta) + B_n\sin(n\theta).)$$

The A_n's & B_n's are then obtained by setting $r=1$ and noting

$$u(1,\theta) = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\Rightarrow A_0 = 1, \quad 17A_2 = -\frac{1}{2} \Rightarrow A_2 = -\frac{1}{34},$$

all other A_n's & B_n's are zero, so

$$u(r,\theta) = \frac{1}{2} - \frac{1}{34}(r^2 + 16r^{-2})\cos(2\theta)$$

(b) If we repeat imposing the b.c. $R(2)=0$ we find

$$R_0(r) = C_0(\ln 2 - \ln r)$$

$$R_n(r) = C_n(r^n - 2^{2n}r^{-n})$$

& solving for the constants by apply the condition on the inner boundary gives

$$u(r,\theta) = \frac{1}{2\ln 2}(\ln 2 - \ln r) + \frac{1}{30}\left[r^2 - 16r^{-2}\right]\cos(2\theta)$$

p.193 Problem 3:

Recall that

$$u(x+h) = u(x) + h u'(x) + \frac{1}{2} h^2 u''(x) + \frac{1}{3!} h^3 u'''(x)$$
$$+ \frac{1}{4!} h^4 u^{(4)}(x) + \frac{1}{5!} h^5 u^{(5)}(x) + O(h^6)$$

Then note that

$$\frac{2}{3} u(x+h) - \frac{2}{3} u(x-h) - \frac{1}{12} u(x+2h) + \frac{1}{12} u(x-2h)$$

$$= \frac{2}{3} \cancel{u(x)} + \frac{2}{3} h u'(x) + \frac{1}{3} h^2 \cancel{u''(x)} + \frac{2}{3 \cdot 3!} h^3 u'''(x) + \frac{2}{3} \frac{1}{4!} \cancel{h^4} u^{(4)}(x)$$
$$+ \frac{2}{3} \frac{1}{5!} h^5 u^{(5)}(x)$$

$$- \frac{2}{3} \cancel{u(x)} + \frac{2}{3} h u'(x) - \frac{1}{3} \cancel{h^2 u''(x)} + \frac{2}{3 \cdot 3!} h^3 u'''(x) - \frac{2}{3} \frac{1}{4!} \cancel{h^4 u^{(4)}(x)}$$
$$+ \frac{2}{3} \frac{1}{5!} h^5 u^{(5)}(x)$$

$$- \frac{1}{12} \cancel{u(x)} - \frac{1}{12} (2h) u'(x) - \frac{1}{12} \cancel{(2h)^2} u''(x) - \frac{1}{12} \frac{1}{3!} (2h)^3 u'''(x)$$
$$- \frac{1}{12} \cancel{(2h)^4} u^{(4)}(x) - \frac{1}{12} \frac{1}{5!} (2h)^5 u^{(5)}(x)$$

$$+ \frac{1}{12} \cancel{u(x)} - \frac{1}{12} (2h) u'(x) + \frac{1}{12} \cancel{(2h)} u''(x) - \frac{1}{12} \frac{1}{3!} (2h)^3 u'''(x)$$
$$+ \frac{1}{12} \cancel{(2h)^4} u^{(5)}(x) - \frac{1}{12} \frac{1}{5!} (2h)^5 u^{(5)}(x) + O(h^6)$$

$$= h u'(x) + \frac{2}{9} \cancel{h^3 u'''(x)} + \frac{1}{90} h^5 u^{(5)}(x)$$
$$- \frac{2}{9} \cancel{h^3 u'''(x)} - \frac{64}{12 \cdot 5 \cdot 12} h^5 u^{(5)}(x) + O(h^6)$$

$$= h u'(x) + O(h^5)$$ so

$$u'(x) = \frac{1}{h} \left(\frac{2}{3} u(x+h) - \frac{2}{3} u(x-h) - \frac{1}{12} u(x+2h) + \frac{1}{12} u(x-2h) \right) + O(h^4)$$

p. 199 Problem 1 (see Mathematica notebook)

problem # 7: From the heat eqn we know

$$\frac{\partial u}{\partial t}(j\Delta x, (n+\tfrac{1}{2})\Delta t) = \frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t)$$

It's easy to show using Taylor's Theorem that

$$\frac{\partial u}{\partial t}(j\Delta x, (n+\tfrac{1}{2})\Delta t) = \frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta t)^2$$

Thus, we concentrate on the error incurred by the approximation on the RHS of (15) on p. 198 with $\theta = \tfrac{1}{2}$.

We need to estimate

$$\left| \tfrac{1}{2}(\delta^2 u)_j^n + \tfrac{1}{2}(\delta^2 u)_j^{n+1} - \frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t) \right|$$

$$= \Big| \left(\tfrac{1}{2}(\delta^2 u)_j^n - \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, n\Delta t)\right) + \left(\tfrac{1}{2}(\delta^2 u)_j^{n+1} - \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+1)\Delta t)\right)$$

$$+ \Big(\tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, n\Delta t) + \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+1)\Delta t)$$

$$- \frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t)\Big) \Big|$$

But by the results on the centered difference approx. to the second derivative that we proved in class we have

$$\left| \left(\tfrac{1}{2}(\delta^2 u)_j^n - \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, n\Delta t)\right)\right| = O(\Delta x)^2$$

$$\left| \left(\tfrac{1}{2}(\delta^2 u)_j^{n+1} - \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+1)\Delta t)\right)\right| = O(\Delta x)^2$$

p.199 problem 7 (cont).

Thus, we only need to consider

$$(*) \quad \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, n\Delta t) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+1)\Delta t) - \frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t).$$

But

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t \pm \tfrac{1}{2}\Delta t) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(j\Delta x, (n+\tfrac{1}{2})\Delta t)$$

$$\pm \frac{1}{4}\Delta t \frac{\partial^3 u}{\partial t \partial x}(j\Delta x, (n+\tfrac{1}{2})\Delta t) + \mathcal{O}(\Delta t)^2$$

Inserting this into (*) all terms cancel except those of $\mathcal{O}(\Delta t)^2$ and hence all the errors in the Crank-Nicolson approx. are either $\mathcal{O}(\Delta x)^2$ or $\mathcal{O}(\Delta t)^2$.

page 200 #10

(a) $$\frac{u_{n+1,m} - u_{n-1,m}}{2\Delta t} = \frac{u_{n,m+1} + u_{n,m-1} - 2u_{n,m}}{(\Delta x)^2}$$

So $$\boxed{u_{n+1,m} = 2s(u_{n,m+1} + u_{n,m-1} - 2u_{n,m}) + u_{n-1,m}}$$

with $s = (\Delta t)/(\Delta x)^2$

- This is an explicit scheme.

(b) If we write $u_{n,m} = T(n)\,X(m)$ we find

$$[T(n+1) - T(n-1)]\,X(m) = 2s\,T(n)\,[X(m+1) + X(m-1) - 2X(m)]$$

or $$\frac{T(n+1) - T(n-1)}{T(n)} = \frac{2s[X(m+1) + X(m-1) - 2X(m)]}{X(m)} = -\lambda$$

We can as usual show that the eigenfunctions of the X-eqn are

$$X^{(k)}(m) = \sin(\theta_k m) \quad \text{with } \theta_k = \frac{\pi k}{M},\ k =$$

& the e'values are

$$\lambda_k = \frac{-2s[\sin(\theta_k(m+1)) + \sin(\theta_k(m-1)) - 2\sin(\theta_k m)]}{\sin(\theta_k m)}$$

$$= \frac{-2s[2\sin(\theta_k m)\cos(\theta_k) - 2\sin(\theta_k m)]}{\sin(\theta_k m)}$$

p. 200 # 10 (cont.)

or $\lambda_k = 4s[1 - \cos\theta_k]$, $k = 1, \ldots, M-1$

N.B. $\lambda_k > 0 \quad \forall\ k = 1, \ldots, M-1$

Now consider the $T(n)$ eqn:

$$T_k(n+1) - T_k(n-1) = -\lambda_k T(n)$$

or $$T_k(n+1) + \lambda_k T(n) - T_k(n-1) = 0.$$

We look for solutions of this eqn of the form

$$T_k(n) = \beta^n \quad \text{for some const } \beta.$$

Then
$$\beta^{n+1} + \lambda_k \beta^n - \beta^{n-1} = 0$$

$$\Leftrightarrow \quad \beta^2 + \lambda_k \beta - 1 = 0 \quad \text{or} \quad \beta_k^{\pm} = \frac{-\lambda_k \pm \sqrt{\lambda_k^2 + 4}}{2}.$$

Note that since $\lambda_k > 0$,

$$\beta_k^- = \frac{-\lambda_k - \sqrt{4 + \lambda_k^2}}{2} < -1$$

Thus, the solution $u_k^-(n,m) = (\beta_k^-)^n \sin(\theta_k m)$ grows without bound no matter how small s is and the scheme is unstable.