Homework Assignment1 (Most problems from Evans) **Math 776**

Due Monday September 23

1. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is C^∞. Prove that

$$f(x) = \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathcal{O}(|x|^{k+1})$$

as $|x| \to 0$. This is Taylor's formula in multi-index notation. You may use Taylor's Theorem for functions on $\mathbb{R}$ in your proof.

2. Find an expression for the solution of the initial value problem

$$\begin{aligned} u_t + b \cdot Du + cu &= 0 \text{ in } \mathbb{R}^n \times (0,\infty) && (1) \\ u &= g \text{ on } \mathbb{R}^n \times t = 0 && (2) \end{aligned}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$. (Hint: Make a change of variables to transform the equation into $v_t + b \cdot Dv = 0$, which we solved in class and where $v(x,t)$ is related to $u(x,t)$ in a simple way.)

3. Prove that there exists a constant $C > 0$, depending only on the dimension n such that any solution, u, of

$$\begin{aligned} -\Delta u &= f \text{ in } B(0,1) && (3) \\ u &= g \text{ on } \partial B(0,1) && (4) \end{aligned}$$

satisfies the estimate

$$\max_{B(0,1)} |u| \le C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|) \ .$$

Hints: You may want to use the Green's function for Laplace's equation in a ball. (See Evan's section 2.2.c.) You may also want to use the linearity of the problem to break the solution into a sum of two pieces, one of which is a homogeneous equation with non-homogeneous boundary conditions and the other an inhomogeneous PDE with homogeneous boundary conditions.

4. Given some open, bounded, connected domain U, with smooth boundary, the Neumann problem for Laplace's equation is to solve

$$\begin{aligned} -\Delta u &= 0 \text{ in } U \\ Du \cdot \hat{\mathbf{n}} &= g \text{ on } \partial U \ , && (5) \end{aligned}$$

where $\hat{\mathbf{n}}$ is the outward pointing normal to the boundary of U.

(a) Show that this problem can have a solution **only** if $\int_{\partial U} g dS = 0$.

(b) If this problem has a solution, is it unique?

5. Consider the initial value problem for the $n = 1$ heat equation. For general initial values $g \in C^1(\mathbf{R}) \cap L^1(\mathbf{R})$, one can show that

$$\|u(\cdot,t)\|_{L^\infty} \leq C/\sqrt{t}\ .$$

Using the representation of the solution $u(x,t)$ in terms of a convolution of the initial conditions with the fundamental solution, show that if the function $G(y) = \int_{-\infty}^{y} g(z)dz$ is in $L^1(\mathbb{R})$, (which implies in particular that $\int_{-\infty}^{\infty} g(y)dy = 0$) and if $\int_{\infty}^{\infty} |x||G(x)|dx$ is finite, this estimate can be improved to

$$\|u(\cdot,t)\|_{L^\infty} \leq C/t\ .$$

6. Extend the class of self-similar solutions we found in the lecture by looking for additional solutions of the one dimensional heat equation of the form

$$u(x,t) = \frac{1}{t^a} w(\frac{x}{\sqrt{t}})$$

Hint: Try to find solutions of this form with $w(\xi) = H(\xi)e^{-\xi^2/4}$, where $H(\xi)$ is a polynomial. The solution we found in the lecture corresponded to the case $H(\xi) = 1$.