From Evans’ textbook, please do the following problems:

1. Write down an explicit formula for a solution of

\[ \begin{align*}
    u_t - \Delta u + cu &= f, \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
    u &= g, \quad \text{on } \mathbb{R}^n \times \{t = 0\},
\end{align*} \tag{1} \]

where \( c \in \mathbb{R} \).

2. (Equipartition of energy). Let \( u \in C^2(\mathbb{R} \times [0, \infty)) \) solve the initial-value problem for the wave equation in one-dimension. Suppose that the initial position of the solution, \( g(x) \), and the initial velocity, \( h(x) \), have compact support. The kinetic energy is

\[ k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) \, dx \]

and the potential energy is

\[ p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) \, dx. \]

Prove that \( k(t) = p(t) \) for all \( t \) sufficiently large. (Can you interpret the meaning of the smallest value \( T \) such that \( k(t) = p(t) \) for all \( t \geq T \)?)

3. Use the method of characteristics to solve the following first order partial differential equations:

(a) \( x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1). \)

(b) \( uu_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = x_1/2. \)

(c) \( x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2). \)

Then solve:

4. Consider the one-dimensional heat-equation on the half-line \( x > 0 \), with a perfectly insulating boundary condition at the origin - i.e. we assume that \( u_x(0, t) = 0 \) for all \( t \geq 0 \). If \( u(x, 0) = g(x) \), for \( x > 0 \), show that one can write the solution of the initial value problem as

\[ u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[ e^{-(x-y)^2/(4t)} + e^{-(x+y)^2/(4t)} \right] g(y) \, dy. \]

5. Let \( \Omega \) by a bounded region in \( \mathbb{R}^2 \) with smooth boundary. The motion of a thin, vibrating plate with shape \( \Omega \) and clamped edges is approximated by the equation

\[ \begin{align*}
    \frac{\partial^2 u}{\partial t^2} &= -\Delta^2 u \\
    u(x, t) &= 0 \quad \text{for } x \in \partial \Omega \\
    Du(x, t) \cdot \hat{n} &= 0 \quad \text{for } x \in \partial \Omega
\end{align*} \]

where \( \hat{n} \) is the outward pointing unit normal vector on the boundary of \( \Omega \). Show that if we specify initial conditions \( u(x, 0) = g(x), \frac{\partial u}{\partial t}(x, 0) = h(x) \), this problem has at most one
solution. (Hint: Try to find a conserved “energy” for this problem.) Note the “bi-Laplacian” operator \( \Delta^2 u = \Delta(\Delta u) = \partial^4_x u + 2\partial^2_x \partial^2_y u + \partial^4_y u \) in dimension two.

6. Suppose that \( f \in L^1(\mathbb{R}) \) and \( g \in C_c^\infty(\mathbb{R}) \). Show that the convolution of \( f \) and \( g \)

\[
 f \ast g(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy = \int_{-\infty}^{\infty} f(z)g(x - z)dz ,
\]

is in \( C^\infty(\mathbb{R}) \).

Hint: Use the dominated convergence theorem to establish that \( f \ast g \in C^1(\mathbb{R}) \) and then use an induction argument.