Homework Assignment 2 **Math 776**
Due Wednesday October 9

From Evans' textbook, please do the following problems:

1. Write down an explicit formula for a solution of

$$\begin{aligned} u_t - \Delta u + cu &= f\ , \quad \text{in } \ \mathbb{R}^n \times (0, \infty) \\ u &= g\ , \quad \text{on } \ \mathbb{R}^n \times \{t = 0\}\ , \end{aligned} \tag{1}$$

where $c \in \mathbb{R}$.

2. (Equipartition of energy). Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one-dimension. Suppose that the initial position of the solution, $g(x)$, and the initial velocity, $h(x)$, have compact support. The *kinetic energy* is $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the *potential energy* is $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Prove that $k(t) = p(t)$ for all t sufficiently large. (Can you interpret the meaning of the smallest value T such that $k(t) = p(t)$ for all $t \geq T$?)

3. Use the method of characteristics to solve the following first order partial differential equations:

(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u,\ \ u(x_1, 1) = g(x_1)$.

(b) $u u_{x_1} + u_{x_2} = 1,\ \ u(x_1, x_1) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u,\ \ u(x_1, x_2, 0) = g(x_1, x_2)$.

Then solve:

4. Consider the one-dimensional heat-equation on the half-line $x > 0$, with a perfectly insulating boundary condition at the origin - *i.e.* we assume that $u_x(0, t) = 0$ for all $t \geq 0$. If $u(x, 0) = g(x)$, for $x > 0$, show that one can write the solution of the initial value problem as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} [e^{-(x-y)^2/(4t)} + e^{-(x+y)^2/(4t)}] g(y) dy\ .$$

5. Let Ω by a bounded region in $\mathbb{R}^2$ with smooth boundary. The motion of a thin, vibrating plate with shape Ω and clamped edges is approximated by the equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= -\Delta^2 u \\ u(x,t) &= 0 \ \text{ for } \ x \in \partial\Omega \\ Du(x,t) \cdot \hat{n} &= 0 \ \text{ for } \ x \in \partial\Omega \end{aligned}$$

where $\hat{n}$ is the outward pointing unit normal vector on the boundary of Ω. Show that if we specify initial conditions $u(x,0) = g(x)$, $\frac{\partial u}{\partial t}(x,0) = h(x)$, this problem has at most one solution. (*Hint:* Try to find a conserved "energy" for this problem.) Note the "bi-Laplacian" operator $\Delta^2 u = \Delta(\Delta u) = \partial_x^4 u + 2\partial_x^2 \partial_y^2 u + \partial_y^4 u$ in dimension two.

6. Suppose that $f \in L^1(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. Show that the convolution of f and g

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(z)g(x-z)dz \ , \tag{2}$$

is in $C^\infty(\mathbb{R})$.
Hint: Use the dominated convergence theorem to establish that $f * g \in C^1(\mathbb{R})$ and then use an induction argument.