Homework Assignment 5 **Math 776**

Due Monday November 25

1. A function $u \in H_0^2(U)$ is a weak solution of the *biharmonic equation*

$$\Delta^2 u = f \quad \text{in } U \tag{1}$$
$$u = \nabla u \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \partial U \tag{2}$$

if

$$\int_U (\Delta u)(\Delta v) dx = \int_U f v dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of this boundary value problem. (Hint: You can use the Poincaré inequality which says that for $u \in H_0^2(U)$, $\int_U (\Delta u)^2 dx \geq C \int_U u^2 dx$, for some constant C.)

2. Assume that U is connected. A function $u \in H^1(U)$ is a weak solution of *Neumann's problem*

$$\Delta u = f \quad \text{in } U \tag{3}$$
$$\nabla u \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \partial U \tag{4}$$

if

$$\int_U (\nabla u) \cdot (\nabla v) dx = \int_U f v dx$$

for all $v \in H^1(U)$. (Note that both u and v are assumed to lie in $H^1(U)$, not $H_0^1(U)$.) Suppose that $f \in L^2(U)$. Prove that Neumann's problem has a unique solution if and only if

$$\int_U f dx = 0 \ .$$

(Hint: This is related to the Fredholm alternative.)

3. Let $U = \{x \in \mathbf{R}^3 \mid |x| < \pi\}$. Show that a necessary condition for $-\Delta u - u = f$ to have a weak solution in $H_0^1(U)$ is that

$$\int_U f(x) \frac{\sin(|x|)}{|x|} dx = 0.$$

Note: You don't have to show that this condition is sufficient - only that it is necessary. (Hint: This is related to the Fredholm alternative.)

Don't overlook the problem on the back of the page.

A problem from **An Introductions to Partial Differential Equations** by Renardy and Rogers.

4. Show that the techniques of this chapter can be used to treat the following boundary value problem for ODE's:

$$y'' + p(x)y' + q(x)y = f(x), \quad y(0) = y(1) = 0 \ .$$

Assume that $p, q \in C^1[0,1]$, and prove that there is a unique (weak) solution if $p' - 2q \geq 0$.